WHICH QUARTIC DOUBLE SOLIDS ARE RATIONAL?

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Abstract. We study the rationality problem for nodal quartic double solids. In particular, we prove that nodal quartic double solids with at most six singular points are irrational, and nodal quartic double solids with at least eleven singular points are rational.

1. Introduction

In this paper, we study double covers of $\mathbb{P}^3$ branched over nodal quartic surfaces. These Fano threefolds are known as *quartic double solids*. It is well-known that smooth threefolds of this type are irrational. This was proved by Tihomirov (see [23, Theorem 5]) and Voisin (see [25, Corollary 4.7(b)]). The same result was proved by Beauville in [2, Exemple 4.10.4] for the case of quartic double solids with one ordinary double singular point (node), by Debarre in [8] for the case of up to four nodes and also for five nodes subject to generality conditions, and by Varley in [24, Theorem 2] for double covers of $\mathbb{P}^3$ branched over special quartic surfaces with six nodes (so-called Weddle quartic surfaces). All these results were proved using the theory of intermediate Jacobians introduced by Clemens and Griffiths in [6]. In [5, §8 and §9], Clemens studied intermediate Jacobians of resolutions of singularities for nodal quartic double solids with at most six nodes in general position.

Another approach to irrationality of nodal quartic double solids was introduced by Artin and Mumford in [1]. They constructed an example of a quartic double solid with ten nodes whose resolution of singularities has non-trivial torsion in the third integral cohomology group, and thus the solid is not stably rational. Recently, Voisin used this example together with her new approach via Chow groups to prove the following result.

Theorem 1.1 ([26, Theorem 0.1])**.** *For any integer $k = 0, \ldots, 7$, a very general nodal quartic double solid with k nodes is not stably rational.*

In spite of its strength, Theorem 1.1 is not easy to apply to particular varieties. This is due to non-explicit generality condition involved. The main goal of this paper is to get rid of this generality condition (at the cost of weakening the assertion and allowing fewer singular points on a quartic double solid). We use intermediate Jacobian theory together with elementary birational geometry to prove the following result.

Theorem 1.2. *A nodal quartic double solid with at most six nodes is irrational.*

Recall that a nodal quartic surface in $\mathbb{P}^3$ can have at most 16 nodes, so that this is also the maximal number of nodes on a quartic double solid. Moreover, Prokhorov proved that every nodal quartic double solid with 15 or 16 nodes is rational (see [17, Theorem 8.1] and [17, Theorem 7.1], respectively). We use his approach to study quartic double solids with many nodes. In particular, we prove the following result.

Theorem 1.3. *A nodal quartic double solid with at least eleven nodes is rational.*

In fact, we prove a stronger assertion. To describe it, let us recall that a variety is said to be $\mathbb{Q}$-factorial if every Weil divisor on it is $\mathbb{Q}$-Cartier. For a nodal variety, this condition is equivalent to the coincidence of Weil and Cartier divisors.

Example 1.4. Let S be a nodal quartic surface in $\mathbb{P}^3$ with homogeneous coordinates x, y, z, t such that S is given by the equation $g^2 = 4xh$, where g and h are some quadric and cubic forms, respectively. Then S is singular at the points given by $x = g = h = 0$. Since we assume that S is nodal, this system of equations gives exactly six points $P_1, \ldots, P_6$. Let $\tau\colon X \to \mathbb{P}^3$ be a double cover branched over S. Then there exists a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\alpha"] \arrow[d, "\beta"'] & V_3 \arrow[d, dashed, "\gamma"] \\ X \arrow[r, "\tau"] & \mathbb{P}^3. \end{tikzcd}$$

Here V_3 is a smooth cubic threefold in $\mathbb{P}^4$ with homogeneous coordinates x, y, z, t, w such that it is given by equation

$$w^2x + wg + h = 0,$$

the map γ is a linear projection from the point $P = [0 : 0 : 0 : 0 : 1]$, the map α is the blow up of this point, and β is the contraction of the proper transforms of six lines on V_3 that pass through P to the singular points of X. Then the image on X of the α-exceptional surface is not a $\mathbb{Q}$-Cartier divisor. In particular, X is not $\mathbb{Q}$-factorial. Note that β is a small resolution of singularities of X at the points $P_1, \ldots, P_6$. Thus, V_3 is singular if and only if $P_1, \ldots, P_6$ are the only singular points of X. On the other hand, V_3 is irrational if and only if it is smooth (see [6, Theorem 13.12]). Thus, X is irrational if and only if it has exactly six singular points.

It turns out that Example 1.4 provides the only construction of irrational nodal quartic double solids that are not $\mathbb{Q}$-factorial. Namely, we prove the following result.

Theorem 1.5. *A non-$\mathbb{Q}$-factorial nodal quartic double solid is rational unless it has exactly six nodes and is described by Example 1.4.*

Note that the $\mathbb{Q}$-factoriality of nodal quartic double solids can be easily verified using the following result of Clemens.

Theorem 1.6 ([5, §3])**.** *Let X be a double cover of $\mathbb{P}^3$ branched over a nodal quartic surface S. Then X is $\mathbb{Q}$-factorial if and only if the nodes of S impose independent linear conditions on quadrics in $\mathbb{P}^3$.*

In particular, this theorem gives the following result that implies Theorem 1.3 by Theorem 1.5.

Corollary 1.7. *Nodal quartic double solids with at least eleven nodes are not $\mathbb{Q}$-factorial.*

On the other hand, nodal quartic double solids with at most five nodes are always $\mathbb{Q}$-factorial. This is implied by the following result.

Theorem 1.8 ([11], [3, Theorem 6])**.** *A nodal quartic double solid with at most seven nodes is $\mathbb{Q}$-factorial unless it has exactly six nodes and is described by Example 1.4.*

Thus, one can generalize our Theorem 1.2 as follows.

Conjecture 1.9. *Every $\mathbb{Q}$-factorial nodal quartic double solid is irrational.*

By Theorem 1.5, this conjecture gives a complete answer to the question in the title of this paper in the case of nodal quartic double solids. Here we show that it follows from Shokurov's famous [21, Conjecture 10.3], see Corollary 5.2. Note that the intermediate Jacobians of resolutions of singularities for nodal quartic double solids with more than six nodes are sums of Jacobians of curves, so that the methods of [6] are not applicable to prove Conjecture 1.9 in this case.

The paper is organized as follows. In Section 2, we review some well-known facts about conic bundles over rational surfaces including their Prym varieties and results concerning irrationality of such threefolds. In Section 3, we show how to birationally transform a nodal $\mathbb{Q}$-factorial singular quartic double solid into a conic bundle and study the singularities of its degeneration curve. In Section 4, we present an explicit birational transformation of the latter conic bundle to a standard one. In Section 5, we prove Theorem 1.2 and show that [21, Conjecture 10.3] implies our Conjecture 1.9. In Section 6, we prove Theorem 1.5. In a sequel [4], we apply Theorems 1.2 and 1.3 to nodal quartic double solids having an icosahedral symmetry.

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Notation and conventions. All varieties are assumed to be algebraic, projective and defined over $\mathbb{C}$. By a *node*, we mean an isolated ordinary double singular point of a variety of arbitrary dimension. A variety is called *nodal* if its only singularities are nodes. By a *cusp*, we mean a plane curve singularity of type $\mathbb{A}_2$. By a *tacnode*, we mean a plane curve singularity of type $\mathbb{A}_3$. By $\mathbb{F}_n$ we denote a Hirzebruch surface $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)\big)$. Given a birational morphism $\varphi\colon X \to Y$ and a linear system $\mathcal{M}$ on Y, by a proper transform (sometimes also called a homaloidal transform) of $\mathcal{M}$ we mean the linear system generated by divisors $\varphi^{-1}M$, where M is a general divisor in $\mathcal{M}$; since a rational map is a morphism in a complement to a closed subset of codimension 2, we will also use this terminology in case when φ is an arbitrary birational map. If $\mathcal{M}$ is base point free and φ is an arbitrary rational map, then the proper transform of $\mathcal{M}$ is defined as a composition of its pull-back via a regularization of φ and a proper transform via the corresponding birational map.

2. Conic bundles over rational surfaces

Let $\nu\colon V \to U$ be a conic bundle such that V is a threefold, and let U be a surface. Recall that ν is said to be *standard* if both V and U are smooth, and the relative Picard

group of V over U has rank 1. It is well-known that there exists a commutative diagram

$$\begin{tikzcd} V' \arrow[r, dashed, "\rho"] \arrow[d, "\nu'"'] & V \arrow[d, "\nu"] \\ U' \arrow[r, dashed, "\varrho"] & U \end{tikzcd}$$

such that ρ and ϱ are birational maps, and ν' is a standard conic bundle (see, for example, [20, Theorem 1.13]). Because of this, we assume here that ν is already standard. In particular, we assume that V and U are both smooth.

In this section, we discuss some obstructions for V to be rational. In particular, we also assume that U is rational, since otherwise irrationality of V is easy to show.

Denote by Δ the degeneration curve of the conic bundle ν. Since ν is assumed to be standard, the curve Δ is nodal (see, for example, [20, Corollary 1.11]). Restricting the conic bundle ν to Δ, taking the normalization of the resulting surface, and considering the Stein factorization of the the induced morphism to Δ, we obtain a nodal curve Δ' together with an involution I on it such that

$$\Delta'/I \cong \Delta,$$

the nodes of Δ' are exactly the fixed points of I, and I does not interchange branches at these points. In particular, the number of connected components of Δ' is the same as that of Δ. Alternatively, one can construct Δ' as a Hilbert scheme of lines in the fibers of ν over Δ.

Corollary 2.1. *The curve Δ satisfies the following conditions:*

(A) *for every splitting $\Delta = \Delta_1 \cup \Delta_2$, the number $|\Delta_1 \cap \Delta_2|$ is even;*
(B) *for every connected component Δ_1 of the curve Δ, one has $\Delta_1 \not\cong \mathbb{P}^1$.*

Proof. Assertion (A) follows from the fact that a double cover of a smooth curve is ramified over an even number of points. Assertion (B) follows from the fact that $\mathbb{P}^1$ does not have connected unramified double covers. □

In [21], Shokurov formulated the following conjecture.

Conjecture 2.2 ([21, Conjecture 10.3])**.** *If $|2K_U + \Delta| \neq \varnothing$, then V is irrational.*

It follows from [2, Théorème 4.9] that this conjecture holds for $U = \mathbb{P}^2$. In [21, §10], Shokurov proved that Conjecture 2.2 holds also for $U = \mathbb{F}_n$.

Remark 2.3. Let Γ be a connected nodal curve. Suppose that there exists a connected nodal curve Γ' together with an involution ι on it such that $\Gamma'/\iota \cong \Gamma$, the nodes of Γ' are exactly the fixed points of ι, and ι does not interchange branches at these points. Then one can construct a principally polarized abelian variety $\mathrm{Prym}(\Gamma', \iota)$ known as the *Prym variety* of the pair (Γ', ι). For details and basic properties of $\mathrm{Prym}(\Gamma', \iota)$, see [2, §0] or [21].

Consider $\mathrm{Prym}(\Delta', I)$. Its importance is due to the following result.

Theorem 2.4 (see [2, Proposition 2.8] and the discussion before it)**.** *Let $\mathrm{J}(V)$ be the intermediate Jacobian of V. Then $\mathrm{J}(V) \cong \mathrm{Prym}(\Delta', I)$ (as principally polarized abelian varieties).*

The dualizing sheaf of the curve Δ is free (see e. g. [10, Exercise 3.4(1)]). Moreover, the linear system $|K_\Delta|$ is base point free by Corollary 2.1. Hence, it gives the canonical morphism

$$\kappa_\Delta\colon \Delta \to \mathbb{P}^N,$$

where $N = h^0(\mathcal{O}_\Delta(K_\Delta)) - 1$. Note that κ_Δ may contract irreducible components of Δ. If Δ is connected, then it is said to be

- *hyperelliptic* if there is a morphism $\Delta \to \mathbb{P}^1$ that has degree two over a general point of $\mathbb{P}^1$;
- *trigonal* if there is a morphism $\Delta \to \mathbb{P}^1$ that has degree three over a general point of $\mathbb{P}^1$;
- *quasitrigonal* if it is a hyperelliptic curve with two glued smooth points.

Remark 2.5. Suppose that Δ is connected. If the curve Δ is hyperelliptic, then $\kappa_\Delta(\Delta)$ is a rational normal curve of degree N, and the induced map $\Delta \to \kappa_\Delta(\Delta)$ has degree two over a general point of $\kappa_\Delta(\Delta)$. If the curve Δ is trigonal, then the curve $\kappa_\Delta(\Delta)$ has trisecants, so that $\kappa_\Delta(\Delta)$ is not an intersection of quadrics. If Δ is quasitrigonal, then the intersection of quadrics passing through $\kappa_\Delta(\Delta)$ is a cone over a rational normal curve of degree $N-1$.

The main result of [21] is the following theorem.

Theorem 2.6 ([21, Main Theorem])**.** *In the notation and assumptions of Remark 2.3, suppose that Γ is connected, and the following condition holds:*

(S) *for any splitting $\Gamma = \Gamma_1 \cup \Gamma_2$, one has $|\Gamma_1 \cap \Gamma_2| \geqslant 4$.*

Then $\mathrm{Prym}(\Gamma', \iota)$ *is a sum of Jacobians of smooth curves if and only if Γ is*

- *either hyperelliptic, or*
- *trigonal, or*
- *quasitrigonal, or*
- *a plane quintic curve such that $h^0(\Gamma, \mathcal{O}_{\mathbb{P}^2}(1)|_\Gamma)$ is odd.*

Thus, Theorems 2.4 and 2.6 imply the following result.

Corollary 2.7. *Suppose that the curve Δ is connected and not hyperelliptic, the curve $\kappa_\Delta(\Delta)$ is an intersection of quadrics in $\mathbb{P}^N$, and condition* (S) *of Theorem 2.6 holds for Δ. Then* $\mathrm{Prym}\,(\Delta', I)$ *is not a sum of Jacobians of smooth curves.*

Proof. By Remark 2.5 and Theorem 2.6, it is enough to show that Δ is not a plane quintic. The latter follows from the fact that quadrics in $\mathbb{P}^5$ that pass through the canonical image C of a plane nodal quintic cut out a Veronese surface, so that C is not an intersection of quadrics. □

Clemens and Griffiths proved in [6, Corollary 3.26] that V is irrational provided that $\mathrm{J}(V)$ is not a sum of Jacobians of smooth curves. Thus, Corollary 2.7 and Theorem 2.4 imply the following result.

Corollary 2.8. *Suppose that the curve Δ is connected and not hyperelliptic, the curve $\kappa_\Delta(\Delta)$ is an intersection of quadrics in $\mathbb{P}^N$, and condition* (S) *of Theorem 2.6 holds for Δ. Then V is irrational.*

Corollary 2.8 and Theorem 2.6 imply Conjecture 2.2 for $U = \mathbb{F}_n$. For details, see the proof of [21, Theorem 10.2].

Remark 2.9. In the notation and assumptions of Remark 2.3, suppose that there is a splitting

$$\Gamma = E_1 \cup \ldots \cup E_r \cup \Phi$$

such that each E_i is a smooth rational curve, the curves $E_1, \ldots, E_r$ are disjoint, and each intersection $E_i \cap \Phi$ consists of two points. Let Θ be a nodal curve obtained from Φ by gluing each pair of points $E_i \cap \Phi$. It follows from [21, Corollary 3.16] and [21, Remark 3.17] that there exists a connected nodal curve Θ' together with an involution σ on it such that

$$\Theta'/\sigma \cong \Theta,$$

the nodes of Θ' are exactly the fixed points of σ, the involution σ does not interchange branches at these points, and

$$\mathrm{Prym}\,(\Gamma', \iota) \cong \mathrm{Prym}\,(\Theta', \sigma)\,.$$

3. Quartic double solids and conic bundles

Let $\tau \colon X \to \mathbb{P}^3$ be a double cover branched over a nodal quartic surface in S. Suppose that S is indeed singular, and let O_S be a singular point of the surface S. Denote by O_X the point in X that is mapped to the point O_S by the double cover τ. Then there exists a commutative diagram

$$\begin{tikzcd}
& X \arrow[r, "\tau"] \arrow[ddr, dashed, "p_{O_X}"] & \mathbb{P}^3 \arrow[dd, dashed, "p_{O_S}"] & S \arrow[l, hook'] \arrow[ddl, dashed, "p_{O_S}|_S"] \\
X_0 \arrow[ur, "f_{O_X}"] \arrow[drr, "\pi"'] & & & \\
& & \mathbb{P}^2 &
\end{tikzcd}$$

where p_{O_S} is the linear projection from the point O_S, the morphism f_{O_X} is the blow up of the point O_X, the map p_{O_X} is undefined only in the point O_X, and π is a conic bundle. One has

$$(\tau \circ f_{O_X})^* \,\mathcal{O}_{\mathbb{P}^3}(1) - E_{O_X} \sim \pi^*\mathcal{O}_{\mathbb{P}^2}(1),$$

where $E_{O_X} \cong \mathbb{P}^1 \times \mathbb{P}^1$ is the exceptional surface of f_{O_X}.

Remark 3.1. The divisor $-K_{X_0}$ is ample and $-K_{X_0}^3 = 14$. If O_X is the only singular point of the surface S, then X_0 is the Fano threefold No. 8 in the notation of [13, §12.3].

The restricted map

$$p_{O_S}|_S \colon S \dashrightarrow \mathbb{P}^2$$

is a generically two-to-one cover, and its branch locus is a curve of degree 6, which is also the degeneration curve of the conic bundle π. Denote this curve by C. The scheme fibers of π over the points of the curve C are singular conics in $\mathbb{P}^2$. Note that the scheme fiber of π over a point $\xi \in \mathbb{P}^2$ is non-reduced if and only if the line L_ξ mapped to ξ by p_{O_X} is contained in the quartic S; in this case one obviously has $\xi \in C$.

Proposition 3.2. *The singularities of the curve C (if any) are nodes, cusps, or tacnodes. Moreover, let ξ be a point in C, and let L_ξ be a line in $\mathbb{P}^3$ that is mapped to ξ by the linear projection p_{O_X}. Then the following assertions hold.*

(i) *The curve C has a tacnode at ξ if and only if $L_\xi \subset S$, and there are exactly two singular points of S different from O_S that are contained in L_ξ.*

(ii) *The curve C has a cusp at ξ if and only if $L_\xi \subset S$, and there is a unique singular point of S different from O_S that is contained in L_ξ.*
(iii) *The curve C has a node at ξ if and only if one of the following two cases holds:*
- *$L_\xi \not\subset S$, and there is a unique singular point of S different from O_S that is contained in L_ξ;*
- *$L_\xi \subset S$, and O_S is the unique singular point of S that is contained in L_ξ.*

Proof. Choose homogeneous coordinates x, y, z, and t in $\mathbb{P}^3$ so that $O_S = [0 : 0 : 1 : 0]$. Then the quartic S is given by equation

$$z^2 q_2(x, y, t) + z q_3(x, y, t) + q_4(x, y, t) = 0, \tag{3.3}$$

where q_i is a form of degree i. One has

$$p_{O_S}([x : y : z : t]) = [x : y : t].$$

Moreover, after a change of coordinates x, y and t we may assume that the line L_ξ is given by equations $x = y = 0$. The equation of the curve C in a local chart $\mathbb{A}^2 \subset \mathbb{P}^2$ with coordinates x and y is written as $F(x, y) = 0$, where

$$F(x, y) = q_3(x, y, 1)^2 - 4q_2(x, y, 1)q_4(x, y, 1). \tag{3.4}$$

Assume that the line L_ξ is not contained in S, i. e. at least one of the forms q_i in (3.3) contains a monomial t^i with non-zero coefficient. Since ξ is contained in C, we see that L_ξ intersects S at O_S and at most one more point.

Suppose that O_S is the only common point of L_ξ and S. Then $q_4(0, 0, 1) \neq 0$ by (3.3). Keeping this in mind and looking at (3.3) once again, we see that

$$q_2(0, 0, 1) = q_3(0, 0, 1) = 0.$$

Since O_S is a node, we also see that at least one of the partial derivatives of q_2 with respect to x and y does not vanish at the point $[0 : 0 : 1]$. We may assume that this is a partial derivative with respect to x. Then (3.4) implies that the partial derivative of F with respect to x at the point $(0, 0)$ does not vanish either, so that C is smooth at the point ξ.

Suppose that the intersection $L_\xi \cap S$ contains a point P_S different from O_S. Making a change of coordinates if necessary, we may assume that

$$P_S = [0 : 0 : 0 : 1].$$

Then $q_4(0, 0, 1) = 0$ by (3.3) and thus $q_3(0, 0, 1) = 0$ by (3.4). This implies $q_2(0, 0, 1) \neq 0$, so that we may assume that $q_2(0, 0, 1) = 1$.

Suppose that P_S is a non-singular point of S. Then at least one of the partial derivatives of q_4 with respect to x and y does not vanish at the point $[0 : 0 : 1]$. As above, we may assume that this is a partial derivative with respect to x. Then (3.4) implies that the partial derivative of F with respect to x at the point $(0, 0)$ does not vanish either, so that C is smooth at ξ.

Suppose that P_S is a singular point of S. Then none of the monomials of q_4 is divisible by t^3. Write

$$q_3(x, y, t) = 2l(x, y)t^2 + \bar{q}_3(x, y, t)$$

and

$$q_4(x, y, t) = q(x, y)t^2 + \bar{q}_4(x, y, t),$$

where every monomial of $\bar{q}_3$ has degree at least 2 in x, y, while every monomial of $\bar{q}_4$ has degree at least 3 in x, y. Using the fact that P_S is a node of S, we conclude that the quadratic form

$$g(x,y) = z^2 + 2l(x,y)z + q(x,y)$$

is non-degenerate. This means that $q(x,y) - l(x,y)^2$ is not a square. On the other hand, we rewrite (3.4) as

$$F(x,y) = 4l(x,y)^2 - 4q(x,y) + F_{\geqslant 3}(x,y),$$

where any monomial of $F_{\geqslant 3}$ has degree at least 3. This implies that the curve C has a node at ξ.

Now assume that the line L_ξ is contained in the quartic S. This means that neither of the forms q_i in (3.3) contains a monomial t^i with non-zero coefficient.

Suppose that L_ξ does not contain singular points of S that are different from O_S. Write

$$q_i(x,y,t) = l_i(x,y)t^{i-1} + \bar{q}_i(x,y,t)$$

for $i = 2,3,4$, where every monomial of $\bar{q}_i$ has degree at least 2 in x, y. Put

$$f(x,y) = l_3(x,y)^2 - 4l_2(x,y)l_4(x,y).$$

One can check that $f(x,y)$ is not a square, and thus is a non-degenerate quadratic form in x and y. On the other hand, we can rewrite (3.4) as

$$F(x,y) = f(x,y) + F_{\geqslant 3}(x,y),$$

where any monomial of $F_{\geqslant 3}$ has degree at least 3. Therefore, the curve C has a node at ξ.

Suppose that L_ξ contains a singular point P_S of S such that P_S is different from O_S. Making a change of coordinates if necessary, we may assume that $P_S = [0:0:0:1]$ and

$$q_2 = xt + y^2. \tag{3.5}$$

Since P_S is a singular point of S, the form q_4 does not contain any of the monomials xt^3 or yt^3 with non-zero coefficient. Since P_S is a node of S, the form q_3 contains at least one of the monomials xt^2 or yt^2 with non-zero coefficient.

Suppose that q_3 contains the monomial yt^2 with non-zero coefficient. It is easy to see from equation (3.3) that in this case O_S and P_S are the only singular points of S contained in L_ξ. Making a further change of coordinates x and y if necessary, we rewrite

$$q_2(x,y,t) = xt + l(x,y)^2, \quad q_3(x,y,t) = yt^2 + \bar{q}_3(x,y,t),$$

and

$$q_4(x,y,t) = \alpha x^2t^2 + \beta xyt^2 + \gamma y^2t^2 + \bar{q}_4(x,y,t),$$

where every monomial of $\bar{q}_3$ has degree at least 2 in x, y, and every monomial of $\bar{q}_4$ has degree at least 3 in x, y. Since P_S is a node of S, we have $\alpha \neq 0$. Assigning the weights $\mathrm{wt}(y) = 3$ and $\mathrm{wt}(x) = 2$, we rewrite (3.4) as

$$F(x,y) = y^2 - 4\alpha x^3 + F_{\geqslant 7}(x,y),$$

where any monomial of $F_{\geqslant 7}$ has weight at least 7. Thus, the curve C has a cusp at the point ξ.

Finally, suppose that q_3 does not contain the monomial yt^2 with non-zero coefficient. It is easy to see from equation (3.3) that in this case there is a unique singular point Q_S of S different from O_S and P_S that is contained in L_ξ. We may write

$$q_3(x,y,t) = \alpha xt^2 + \epsilon y^2t + \bar{q}_3(x,y,t), \tag{3.6}$$

where every monomial of $\bar{q}_3$ has degree at least 2 in x, y, and is different from y^2t. Since P_S is a node of S, we have $\alpha \neq 0$. It is easy to see from equation (3.3) that in this case the unique singular point of S different from O_S and P_S that is contained in L_ξ is

$$Q_S = [0 : 0 : -\alpha : 1].$$

Write

$$q_4(x, y, t) = \beta y^2t^2 + \gamma xyt^2 + \delta x^2t^2 + \bar{q}_4(x, y, t), \tag{3.7}$$

where $\beta \neq 0$, and every monomial of $\bar{q}_4$ has degree at least 3 in x, y. Using once again the fact that P_S is a node of S, we see that $\beta \neq 0$. Choosing a new coordinate $z' = z + \alpha t$, we rewrite (3.3) as

$$\big(\alpha^2 - \alpha\epsilon + \beta\big)\, y^2t^2 + \Phi(x, y, z', t) = 0,$$

where every monomial of Φ either is divisible by x or has degree at least 3 in x, y and z'. Hence the fact that Q_S is a node of S implies that

$$\alpha^2 - \alpha\epsilon + \beta \neq 0.$$

On the other hand, assigning the weights $\mathrm{wt}(x) = 2$ and $\mathrm{wt}(y) = 1$, and using equations (3.5), (3.6) and (3.7), we rewrite (3.4) as

$$F(x, y) = F_4(x, y) + F_{\geqslant 5}(x, y),$$

where

$$F_4(x, y) = \alpha^2x^2 + (2\alpha\epsilon - 4\beta)xy^2 + (\epsilon^2 - 4\beta)y^4,$$

and every monomial of $F_{\geqslant 5}$ has weight at least 5. It is straightforward to check that the polynomial F_4 is not a square. This means that the curve C has a tacnode at ξ. □

Remark 3.8. If O_S is the only singular point of the surface S, then Proposition 3.2 follows from a much more general [20, Corollary 1.11].

Lemma 3.9. *In the notation of Proposition 3.2, suppose that the curve C has a tacnode at the point ξ. Let F_ξ be the preimage of the point ξ with respect to π. Let T be the line in $\mathbb{P}^2$ that passes through ξ and has a local intersection number 4 with the curve C at ξ. Denote by $\mathcal{B}$ the linear system of conics in $\mathbb{P}^2$ that are tangent to T at ξ. Let B_1 and B_2 be preimages on X_0 of two general conics in $\mathcal{B}$. Then each B_i has a singularity locally isomorphic to a product of a node and $\mathbb{A}^1$ at a general point of F_ξ, and*

$$\mathrm{mult}_{F_\xi}(B_1 \cdot B_2) = 4.$$

Proof. Using coordinates in $\mathbb{P}^3$ and $\mathbb{P}^2$ introduced in the proof of Proposition 3.2, we find that the line T is given by equation $x = 0$. Regarding x and y as local coordinates in an affine chart containing ξ, and making an analytic change of coordinates if necessary, we write an equation of a general conic in $\mathcal{B}$ as

$$x - \lambda y^2 = 0,$$

where $\lambda \in \mathbb{C}$.

Keeping in mind equation (3.3), we write down the local equation of X (and also of X_0 at a general point of F_ξ) in $\mathbb{A}^4$ as

$$w^2 = q_2(x, y, t) + q_3(x, y, t) + q_4(x, y, t).$$

Using equations (3.5), (3.6) and (3.7), we see that the surfaces B_i are locally defined by equations

$$w^2 = \mu_i y^2 + F_i(y, t)$$

in local coordinates w, y and t, where μ_i are (different) non-zero constants, and every monomial of F_i has degree at least 3. Since F_ξ is given by $w = y = 0$ in the same coordinates, the assertion of the lemma follows. □

4. From non-standard to standard conic bundles

Let us use all notation and assumptions of Section 3. If S is smooth away of O_S, then the conic bundle $\pi\colon X_0 \to \mathbb{P}^2$ is standard by Theorem 1.8. Moreover, it follows from [20, Theorem 1.13] that there exists a commutative diagram

$$\begin{tikzcd} V \arrow[r, dashed, "\rho"] \arrow[d, "\nu"'] & X_0 \arrow[d, "\pi"] \\ U \arrow[r, "\varrho"] & \mathbb{P}^2, \end{tikzcd} \tag{4.1}$$

where V is a smooth projective threefold, U is a smooth surface, ν is a standard conic bundle, ρ is a birational map, and ϱ is a birational morphism. Of course, (4.1) is not unique. The goal of this section is to explicitly construct (4.1) with ϱ being a composition of $|\mathrm{Sing}(S)| - 1$ blow ups of smooth points. Namely, we prove the following theorem.

Theorem 4.2. *Suppose that X is $\mathbb{Q}$-factorial. Then there exists a commutative diagram (4.1) where ν is a standard conic bundle and the following properties hold.*

(i) *The birational morphism ϱ is a composition of $|\mathrm{Sing}(S)| - 1$ blow ups of smooth points.*

(ii) *The birational morphism ϱ factors as*

$$U \xrightarrow{\varrho_t'} U_t \xrightarrow{\varrho_t} U_c \xrightarrow{\varrho_c} U_n \xrightarrow{\varrho_n} \mathbb{P}^2,$$

where the morphism ϱ_n is a blow up of the nodes of the curve C that are images of the singular points of X_0 via π, the morphism ϱ_c is a blow up of all cusps of the proper transform of C on the surface U_n, the morphism ϱ_t is a blow up of all tacnodes of the proper transform of C on the surface U_c, and the morphism ϱ_t' is a blow up of all nodes of the proper transform of C on the surface U_t that are mapped to the tacnodes of the curve C by $\varrho_n \circ \varrho_c \circ \varrho_t$. In particular, the birational map ϱ^{-1} is regular away of $\mathrm{Sing}(C)$.

(iii) *Let Δ be the degeneration curve of the conic bundle ν. Then Δ is the proper transform of the curve C, i. e. the exceptional curves of ϱ are not contained in Δ. In particular, one has $\Delta \sim -2K_U$.*

In the rest of the section, we will prove Theorem 4.2. Namely, we will show how to construct the commutative diagram (4.1) by analyzing the geometry of X_0 in a neighborhood of a fiber containing a singular point of X_0, producing a desired transformation in such neighborhood, and then applying these constructions together to obtain a global picture.

Let ξ be a point of C, and let F_ξ be the preimage of the point ξ via π. Let L_ξ be a line in $\mathbb{P}^3$ that is mapped to ξ by the linear projection p_{O_X}, so that F_ξ is the preimage of L_ξ via $\tau \circ f_{O_X}$.

Choose homogeneous coordinates x, y, z and t in $\mathbb{P}^3$ so that $O_S = [0:0:1:0]$ and the line L_ξ is given by equations $x = y = 0$. One has

$$p_{O_S}([x:y:z:t]) = [x:y:t].$$

During our next steps we will always assume that the quartic S is singular at some point P_S of the line L_ξ such that P_S is different from O_S; we can choose x, y, z and t so that

$$P_S = [0:0:0:1].$$

Since P_S is a node of S, we know that S is given by equation

$$t^2 q_2(x,y,z) + t q_3(x,y,z) + q_4(x,y,z) = 0, \tag{4.3}$$

where q_i is a form of degree i in three variables, and the quadratic form q_2 is non-degenerate. We can expand (4.3) as

$$\begin{aligned} t^2\big(\alpha z^2 + z q_2^{(1)}(x,y) + q_2^{(2)}(x,y)\big) + \\ + t\big(z^2 q_3^{(1)}(x,y) + z q_3^{(2)}(x,y) + q_3^{(3)}(x,y)\big) + \\ + \big(z^2 q_4^{(2)}(x,y) + z q_4^{(3)}(x,y) + q_4^{(4)}(x,y)\big) = 0, \end{aligned} \tag{4.4}$$

where $q_i^{(j)}$ is a form of degree j in two variables, and α is a constant.

In what follows we will frequently use the following easy and well known auxiliary result.

Lemma 4.5. *Let Y be a normal threefold, R be a surface in Y, and L be a smooth rational curve in R such that R and Y are smooth along L. Suppose that $\mathcal{N}_{L/R} \cong \mathcal{O}_{\mathbb{P}^1}(r)$ and $-K_Y \cdot L = s$ with $2r \geqslant s-2$. Then*

$$\mathcal{N}_{L/Y} \cong \mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(s-r-2).$$

Proof. We have

$$\deg \mathcal{N}_{L/Y} = 2g(L) - 2 - K_Y \cdot L = s-2.$$

Also, there is an injective morphism $\mathcal{N}_{L/R} \hookrightarrow \mathcal{N}_{L/Y}$. Therefore, there is an exact sequence of sheaves on $L \cong \mathbb{P}^1$

$$0 \to \mathcal{O}_{\mathbb{P}^1}(r) \to \mathcal{N}_{L/Y} \to \mathcal{O}_{\mathbb{P}^1}(s-r-2) \to 0.$$

Since $r \geqslant s-r-2$, the latter exact sequence splits and gives the assertion of the lemma. $\square$

Now we are ready to describe birational maps that are needed to transform π to a standard conic bundle.

Construction I. Suppose that S is singular at exactly two points O_S and P_S of the line L_ξ, and L_ξ is not contained in S. This happens if and only if one has $\alpha \neq 0$ in equation (4.4). In particular, we can assume that

$$q_2(x,y,z) = xy + z^2. \tag{4.6}$$

Denote by P_0 the preimage of the point P_S on X_0. The threefold X_0 has a node at P_0 and is smooth elsewhere along F_ξ. The fiber F_ξ consists of two smooth rational curves that intersect transversally at the point P_0.

Let $f_{P_0}\colon X_1 \to X_0$ be the blow up of the point P_0, and let E_1 be the exceptional divisor of f_{P_0}. One has $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the threefold X_1 is smooth along the proper transform of F_ξ.

Denote by L_0^+ and L_0^- the irreducible components of F_ξ, and denote by L_1^+ and L_1^- their proper transforms on X_1. Then the curves L_1^+ and L_1^- are disjoint smooth rational curves.

Lemma 4.7. *Let L_1 be one of the curves L_1^+ and L_1^-. Then $\mathcal{N}_{L_1/X_1} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.*

Proof. Let $\Pi \subset \mathbb{P}^3$ be a general plane containing the line L_ξ. Then Π is given by

$$\lambda x + \mu y = 0$$

for some $[\lambda : \mu] \in \mathbb{P}^1$. Let R be the preimage of Π via τ. We see from equation (4.4) that R has nodes at the preimages of the points P_S and O_S and is smooth elsewhere.

Let R_1 be the proper transform of R on the threefold X_1. Then the surface R_1 is smooth. One has $K_{R_1} \cdot L_1 = -1$, so that $L_1^2 = -1$ on R_1, and the normal bundle

$$\mathcal{N}_{L_1/R_1} \cong \mathcal{O}_{\mathbb{P}^1}(-1).$$

On the other hand, we know that

$$K_{X_1} \cdot L_1 = 0,$$

which implies the assertion by Lemma 4.5. $\square$

By Lemma 4.7 one can flop each of the curves L_1^+ and L_1^-. Namely, each of these two flops is just an Atiyah flop, i. e. it can be obtained by blowing up the curve L_1^+ or L_1^- and blowing down the exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ along another ruling onto a curve contained in a smooth locus of the resulting threefold (see [16, §4.2], [12, §2]). Let $\chi \colon X_1 \dashrightarrow X_2$ be the composition of Atiyah flops in the curves L_1^+ and L_1^-. Let $f_\xi \colon U_2 \to \mathbb{P}^2$ be the blow up of the point ξ, and $p_2 \colon X_2 \dashrightarrow U_2$ be the corresponding rational map. Put $p_1 = p_2 \circ \chi$.

Let $Z \cong \mathbb{P}^1$ be the exceptional divisor of the blow up f_ξ, and C_2 be the proper transform of the curve C on U_2. By Proposition 3.2(iii) the intersection $C_2 \cap Z$ consists of two points, and C_2 is smooth at these points. Let E_2 be the proper transform of the divisor E_1 on the threefold X_2.

Lemma 4.8. *The rational map p_2 is a morphism, and $p_2(E_2) = Z$. The fiber of p_2 over each of the two points in $C_2 \cap Z$ is a union of two smooth rational curves that intersect transversally at one point. All other fibers of p_2 over Z are smooth, so that C_2 is the degeneration curve of the conic bundle p_2.*

Proof. Denote by $\omega_1 \colon W \to X_1$ the blow up of X_1 along the curves L_1^+ and L_1^-, so that there is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\omega_2"'] \arrow[dr, "\omega_1"] & \\ X_2 & & X_1 \arrow[ll, dashed, "\chi"'] \end{tikzcd}$$

Let G_W^+ and G_W^- be the exceptional divisors of ω_1 over the curves L_1^+ and L_2^+, respectively. Recall that

$$G_W^+ \cong G_W^- \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Denote by l_1^+ and l_1^- the classes of the rulings of G_W^+ and G_W^- that are mapped surjectively onto L_1^+ and L_1^- by ω_1 (and are contracted by ω_2), and denote by l_2^+ and l_2^- the classes of the rulings of G_W^+ and G_W^- that are contracted by ω_1. Let E_W^P be the proper transforms of the surface E_1 on W. One has

$$E_W^P|_{G_W^+} \sim l_2^+, \quad E_W^P|_{G_W^-} \sim l_2^-.$$

Let $\mathcal{H}$ be the pencil of curves that are proper transforms on U_2 of lines in $\mathbb{P}^2$ passing through the point ξ. Note that the class of $H + R$, where $H \in \mathcal{H}$ and $R \in |f_\xi^*\mathcal{O}_{\mathbb{P}^2}(1)|$, is very ample. Note also that the proper transform on X_2 of the linear system $|f_\xi^*\mathcal{O}_{\mathbb{P}^2}(1)|$ is

base point free. Thus, to conclude that the rational map p_2 is a morphism it is enough to check that the proper transform $\mathcal{H}_{X_2}$ of the linear system $\mathcal{H}$ on X_2 has no base points.

Let us first show that the proper transform $\mathcal{H}_W$ of the pencil $\mathcal{H}_{X_2}$ on W is base point free. By construction, its base locus is contained in the union $G_W^+ \cup G_W^- \cup E_W^P$. One has

$$\mathcal{H}_W \sim (\pi \circ f_{P_0} \circ \omega_1)^* \mathcal{O}_{\mathbb{P}^2}(1) - G_W^+ - G_W^- - E_W^P.$$

This gives $\mathcal{H}_W|_{G_W^+} \sim l_1^+$ and $\mathcal{H}_W|_{G_W^-} \sim l_1^-$. Therefore, either two different elements of the pencil $\mathcal{H}_W$ do not have intersection points in G_W^+, or all of them contain one and the same ruling of class l_1^+. The latter is impossible since the proper transforms of elements of $\mathcal{H}$ on X_1 are transversal to each other at a general point of L_1^+. Thus, $\mathcal{H}_W$ does not have base points on G_W^+. In a similar way we see that it does not have base points on G_W^-.

Let us check that the pencil $\mathcal{H}_W$ has no base points in E_W^P. It is most convenient to do this by analyzing the behavior of the rational map p_1 along the surface E_1. Using equation (4.6) and writing down the equation of X, we see that the surface E_1 is identified with a quadric surface given by

$$xy + z^2 = w^2$$

in $\mathbb{P}^3$ with homogeneous coordinates x, y, z, and w. Note that x and y can be interpreted as homogeneous coordinates on Z. The closure of the image of E_1 with respect to the rational map p_1 is the curve Z. The restriction p_{E_1} of p_1 to E_1 is given by

$$[x : y : z : w] \mapsto [x : y]. \tag{4.9}$$

Therefore, p_{E_1} is a projection from the line $x = y = 0$, which intersects E_1 at the points $[0 : 0 : 1 : 1]$ and $[0 : 0 : 1 : -1]$. Note that these are the points $P_1^+ = L_1^+ \cap E_1$ and $P_1^- = L_1^- \cap E_1$ up to permutation. This implies that the pencil $\mathcal{H}_W$ has no base points in E_W^P except possibly in the two curves contracted to P_1^+ and P_1^- by ω_1. But these curves are contained in the divisors G_W^+ and G_W^-, respectively, and we already know that $\mathcal{H}_W$ has no base points in these surfaces. Thus, the pencil $\mathcal{H}_W$ is base point free. In particular, we see that $p_2 \circ \omega_2$ is a morphism.

The restrictions of $\mathcal{H}_W$ to the surfaces G_W^+ and G_W^- are contained in the fibers of the contraction ω_2. This shows that the pencil $\mathcal{H}_{X_2}$ is also base point free, so that p_2 is a morphism.

The remaining assertions of the lemma follow from (4.9). □

Putting everything together, we obtain a commutative diagram

$$\begin{tikzcd}
& W \arrow[dl, "\omega_2"'] \arrow[dr, "\omega_1"] & & \\
X_2 \arrow[dd, "p_2"'] & & X_1 \arrow[ll, dashed, "\chi"'] \arrow[ddll, dashed, "p_1"] \arrow[dr, "f_{P_0}"] & \\
& & & X_0 \arrow[d, "\pi"] \\
U_2 \arrow[rrr, "f_\xi"] & & & \mathbb{P}^2,
\end{tikzcd}$$

Construction II. Suppose that S is singular at exactly two points O_S and P_S of the line L_ξ, and L_ξ is contained in S. This happens if and only if in equation (4.4) one has $\alpha = 0$, and the linear forms $q_2^{(1)}(x,y)$ and $q_3^{(1)}(x,y)$ are not proportional. In particular, we can assume that $q_2^{(1)}(x,y) = x$ and

$$q_2(x,y,z) = xz + y^2. \tag{4.10}$$

As in Construction I, denote by P_0 the preimage on X_0 of the point P. Note that F_ξ is a smooth rational curve passing through P_0. The threefold X_0 has a node at P_0 and is smooth elsewhere along F_ξ.

Let $f_{P_0}\colon X_1 \to X_0$ be the blow up of the point P_0, and let E_1 be the exceptional divisor of f_{P_0}. One has $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Denote by L_1 the proper transform of F_ξ on X_1. The threefold X_1 is smooth along L_1.

We need the following auxiliary result which is actually easy and well known.

Lemma 4.11. *Let Y be a normal threefold, and C be a smooth rational curve contained in the smooth locus of Y. Let P be a point on C, and $h\colon Y' \to Y$ be the blow up of P. Let C' be the proper transform of C on Y'. Write $\mathcal{N}_{C'/Y'} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$. Then*

$$\mathcal{N}_{C/Y} \cong \mathcal{O}_{\mathbb{P}^1}(a+1) \oplus \mathcal{O}_{\mathbb{P}^1}(b+1).$$

Proof. Suppose that $\mathcal{N}_{C/Y} \cong \mathcal{O}_{\mathbb{P}^1}(c) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$. One has

$$(d+c) - (a+b) = \deg \mathcal{N}_{C/Y} - \deg \mathcal{N}_{C/Y} = 2. \tag{4.12}$$

Let $g\colon W \to Y$ be the blow up of the curve C, and G be the exceptional divisor of g. Then G is a Hirzebruch surface $\mathbb{F}_r$, where $r = |c-d|$. Let Z be the fiber of the projection $g|_G\colon G \to C$ over the point P. Let $g'\colon W' \to Y'$ be the blow up of the curve C', and G' be the exceptional divisor of g'. Then G' is a Hirzebruch surface $\mathbb{F}_{r'}$, where $r' = |a-b|$.

Note that there is a morphism $h'\colon W' \to W$ that is a blow up of the curve Z. It gives the following commutative diagram:

$$\begin{tikzcd} & W' \arrow[dl, "g'"'] \arrow[dr, "h'"] & \\ Y' \arrow[dr, "h"'] & & W \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

In particular, the surface G' is the proper transform of the surface G with respect to h', so that $G \cong G'$. Thus we have

$$|c-d| = r = r' = |a-b|,$$

and applying (4.12) we obtain the assertion of the lemma. $\square$

As in the proof of Lemma 4.7, let $\Pi \subset \mathbb{P}^3$ be a general plane containing the line L_ξ, and let R be the preimage of Π via τ. Denote by R_0 the proper transform of R on the threefold X_0. Then it follows from equation (4.4) that R_0 has a node at the point P_0, one more node at some point $P_\Pi \in F_\xi$, and is smooth elsewhere. One has

$$K_{R_0} \cdot F_\xi = K_{X_0} \cdot F_\xi = -1.$$

Lemma 4.13. *One has $\mathcal{N}_{L_1/X_1} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.*

Proof. Let $f\colon X_1' \to X_1$ be the blow up of the preimage on X_1 of the point P_Π. Put $f' = f_{P_0} \circ f$, so that $f'\colon X_1' \to X_0$ is the blow up of the points P_0 and P_Π. Let R_1' be the proper transform of R_0 on the threefold X_1'. Then the surface R_1' is smooth. Note that the morphism

$$f'|_{R_1'}\colon R_1' \to R_0$$

is the blow up of nodes of R_0, and thus it is crepant. Let L_1' be the proper transform of F_ξ (or L_1) on X_1'. Let E_1' be the exceptional divisor of f' over the point P_0 (i. e. the proper transform on X_1' of the exceptional divisor of f_{P_0}), and E' be the exceptional divisor of f' over the point P_Π.

One has $K_{R_1'} \cdot L_1' = -1$, so that $L_1'^2 = -1$, and the normal bundle $\mathcal{N}_{L_1'/R_1'} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. On the other hand, we know that

$$K_{X_1'} \cdot L_1' = \left(f'^* K_{X_0} + E_1' + 2E'\right) \cdot L_1' = 2,$$

which gives

$$\mathcal{N}_{L_1'/X_1'} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3).$$

by Lemma 4.5. Now the assertion follows from Lemma 4.11. □

Let $f_1\colon \bar{X}_1 \to X_1$ be the blow up of the curve L_1, and let $\bar{G}_1$ be its exceptional surface. By Lemma 4.13 we have $\bar{G}_1 \cong \mathbb{F}_2$. Denote by $\bar{L}_1$ the unique smooth rational curve in $\bar{G}_1$ such that $\bar{L}_1^2 = -2$.

Lemma 4.14. *One has* $\mathcal{N}_{\bar{L}_1/\bar{X}_1} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Proof. Let $\bar{R}_1$ and $\bar{E}_1$ be the proper transforms on $\bar{X}_1$ of the surfaces R and E_1, respectively. Let $\bar{E}_0$ be the proper transform of the exceptional divisor of the blow up $f_{O_X}\colon X_0 \to X$ on $\bar{X}_1$. Denote by l the class of the fiber of the natural projection $\bar{G}_1 \to L_1 \cong \mathbb{P}^1$ in $\mathrm{Pic}(\bar{G}_1)$. Then

$$\bar{E}_0|_{\bar{G}_1} \sim \bar{E}_1|_{\bar{G}_1} \sim l.$$

We compute

$$\bar{R}_1 + \bar{E}_0 + \bar{E}_1 + \bar{G}_1 \sim_{\mathbb{Q}} -\frac{1}{2}\big(f_{O_X} \circ f_0 \circ f_1\big)^* K_X \sim_{\mathbb{Q}} -\frac{1}{2}\left(K_{\bar{X}_1} - \bar{E}_0 - \bar{E}_1 - \bar{G}_1\right).$$

Therefore, one has $\bar{R}_1|_{\bar{G}_1} \sim_{\mathbb{Q}} \bar{L}_1 + l$.

On the other hand, we know that the proper transform of R on X_1 is a del Pezzo surface with a unique node on the curve L_1. Thus, we conclude that $\bar{R}_1|_{\bar{G}_1} = L + T$, where L is a fiber of the projection $\bar{G}_1 \to L_1$, and T is some effective one-cycle such that $T \sim_{\mathbb{Q}} \bar{L}_1$. Hence we have $T = \bar{L}_1$.

Since $f_0 \circ f_1|_{\bar{R}_1}\colon \bar{R}_1 \to R_0$ is the minimal resolution of singularities of a nodal del Pezzo surface R_0, we have $K_{\bar{R}_1} \cdot \bar{L}_1 = -1$. Therefore, one has

$$\mathcal{N}_{\bar{L}_1/\bar{R}_1} \cong \mathcal{O}_{\mathbb{P}^1}(-1).$$

Finally, we have $K_{\bar{X}_1} \cdot \bar{L}_1 = 0$, so that the assertion follows by Lemma 4.5. □

By Lemma 4.14 one can make an Atiyah flop $\psi\colon \bar{X}_1 \dashrightarrow \bar{X}_2$ in the curve L_1. Let $\bar{G}_2$ be the proper transform of the surface $\bar{G}_1$ on the threefold $\bar{X}_2$. Then $\bar{G}_2 \cong \mathbb{F}_2$ and there

exists a contraction $f_2\colon \bar{X}_2 \to X_2$ of $\bar{G}_2$ onto a curve contained in the smooth locus of X_2. We have the following commutative diagram:

$$\begin{tikzcd} \bar{X}_2 \arrow[d, "f_2"'] & \bar{X}_1 \arrow[l, dashed, "\psi"'] \arrow[d, "f_1"] \\ X_2 & X_1. \arrow[l, dashed, "\chi"'] \end{tikzcd} \tag{4.15}$$

Note that χ is a flop, and L_1 is a (-2)-curve of width 2 in the notation of [18, Definition 5.3]. The diagram (4.15) is an example of a *pagoda* described in [18, 5.7].

Let $f_\xi\colon U_2 \to \mathbb{P}^2$ be the blow up of the point ξ, and $p_2\colon X_2 \dashrightarrow U_2$ be the corresponding rational map. Put $p_1 = p_2 \circ \chi$. Let $Z \cong \mathbb{P}^1$ be the exceptional divisor of the blow up f_ξ, and C_2 be the proper transform of the curve C on U_2. By Proposition 3.2(ii) the intersection $C_2 \cap Z$ consists of a single point, and C_2 is smooth at this point. Let E_2 be the proper transform of the divisor E_1 on the threefold X_2.

Now we will prove a result that is identical to Lemma 4.8 (but takes place in the setup of our current Construction II).

Lemma 4.16. *The rational map p_2 is a morphism, and $p_2(E_2) = Z$. The fiber of p_2 over the point $C_2 \cap Z$ is a union of two smooth rational curves that intersect transversally at one point. All other fibers of p_2 over Z are smooth, so that C_2 is the degeneration curve of the conic bundle p_2.*

Proof. Denote by $\omega_1\colon W \to \bar{X}_1$ the blow up of $\bar{X}_1$ along the curve $\bar{L}_1$, so that there is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\omega_2"'] \arrow[dr, "\omega_1"] & \\ \bar{X}_2 & & \bar{X}_1 \arrow[ll, dashed, "\psi"'] \end{tikzcd}$$

Let G_W be the exceptional divisor of ω_1. Recall that $G_W \cong \mathbb{P}^1 \times \mathbb{P}^1$. Denote by l_1 the class of the ruling of G_W that is mapped surjectively onto $\bar{L}_1$. Let E_W^P and $G_{1,W}$ be the proper transforms on W of the surfaces E_1 and $\bar{G}_1$, respectively.

As in the proof of Lemma 4.8, let $\mathcal{H}$ be the pencil of the curves that are proper transforms on U_2 of lines in $\mathbb{P}^2$ passing through the point ξ. To show that the rational map p_2 is a morphism, it is enough to check that the proper transform $\mathcal{H}_{X_2}$ of the linear system $\mathcal{H}$ on X_2 is base point free. Let us show first that its proper transform $\mathcal{H}_W$ on the threefold W is base point free.

By construction, we know that all base points of the pencil $\mathcal{H}_W$ are contained in the union $G_W \cup G_{1,W} \cup E_W^P$. One has

$$\mathcal{H}_W \sim (\pi \circ f_{P_0} \circ f_1 \circ \omega_1)^* \mathcal{O}_{\mathbb{P}^2}(1) - G_W - G_{1,W} - E_W^P.$$

This gives $\mathcal{H}_W|_{G_W} \sim l_1$. Therefore, either two different elements of the pencil $\mathcal{H}_W$ do not have intersection points in G_W, or all of them contain one and the same ruling of class l_1. The latter case is impossible; indeed, the proper transforms of elements of $\mathcal{H}$ on $\bar{X}_1$ are tangent to each other along $\bar{L}_1$ with multiplicity 2 since τ is a double cover and the proper transforms of the elements of $\mathcal{H}$ on $\mathbb{P}^3$ are planes passing through the line L_ξ. Therefore, $\mathcal{H}_W$ has no base points in G_W.

Let t_1 be the class of the ruling of $G_{1,W} \cong \mathbb{F}_2$. Then

$$\mathcal{H}_W|_{G_{1,W}} \sim t_1,$$

and the rulings of $G_{1,W}$ cut out by the members of the pencil $\mathcal{H}_W$ vary (cf. the proof of Lemma 4.14). Therefore, $\mathcal{H}_W$ has no base points in $G_{1,W}$.

Let us check that $\mathcal{H}_W$ has no base points in E_W^P by analyzing the behavior of the rational map p_1 along the surface E_1. Using equation (4.10) and writing down the equation of X, we see that the surface E_1 is identified with a quadric surface given by

$$xz + y^2 = w^2$$

in $\mathbb{P}^3$ with homogeneous coordinates x, y, z, and w. Note that x and y can be interpreted as homogeneous coordinates on Z. The closure of the image of E_1 with respect to the rational map p_1 is the curve Z. The restriction p_{E_1} of p_1 to E_1 is given by the formula (4.9). Therefore, p_{E_1} is a projection from the line $x = y = 0$, which is tangent to E_1 at the point $[0 : 0 : 1 : 0]$. Note that this is the point $P_1 = L_1 \cap E_1$. This implies that $\mathcal{H}_W$ has no base points in E_W^P outside the curves contracted to P_1 by $f_1 \circ \omega_1$. But these are exactly the curves $G_{1,W} \cap E_W^P$ and $G_W \cap E_W^P$. Since we already know that $\mathcal{H}_W$ has no base points in $G_{1,W}$ and G_W, we conclude that $\mathcal{H}_W$ is base point free. In particular, the rational map $p_2 \circ f_2 \circ \omega_2$ is a morphism.

Since $\mathcal{H}_W|_{G_W} \sim l_1$, the proper transform $\mathcal{H}_{\bar{X}_2}$ of the pencil $\mathcal{H}_{X_2}$ on the threefold $\bar{X}_2$ is also base point free. Let t_2 be the class of the ruling of $\bar{G}_2 \cong \mathbb{F}_2$. Then

$$\mathcal{H}_{\bar{X}_2}|_{\bar{G}_2} \sim t_2,$$

so that the restriction of $\mathcal{H}_{\bar{X}_2}$ to $\bar{G}_2$ lies in the fibers of the morphism f_2. Therefore, the pencil $\mathcal{H}_{X_2}$ is also base point free, so that p_2 is a morphism.

The remaining assertions of the lemma follow from (4.9). □

Putting everything together, we obtain a commutative diagram

$$\begin{tikzcd}
 & W \arrow[dl, "\omega_2"'] \arrow[dr, "\omega_1"] & & \\
\bar{X}_2 \arrow[d, "f_2"'] & & \bar{X}_1 \arrow[ll, dashed, "\psi"'] \arrow[d, "f_1"] & \\
X_2 \arrow[dd, "p_2"'] & & X_1 \arrow[ll, dashed, "\chi"'] \arrow[ddll, dashed, "p_1"] \arrow[dr, "f_{P_0}"] & \\
 & & & X_0 \arrow[d, "\pi"] \\
U_2 \arrow[rrr, "f_\xi"] & & & \mathbb{P}^2
\end{tikzcd}$$

Construction III. Suppose that S is singular at exactly three points of the line L_ξ, namely O_S, P_S, and some other point Q_S different from O_S and P_S; in particular, this implies that L_ξ is contained in S. This happens if and only if in equation (4.4) one has $\alpha = 0$, and the linear forms $q_2^{(1)}(x, y)$ and $q_3^{(1)}(x, y)$ are proportional. In particular, we can assume that $q_2(x, y, z)$ is given by equation (4.10).

Denote by P_0 and Q_0 the preimages of the points P_S and Q_S on X_0. Note that F_ξ is a smooth rational curve passing through P_0 and Q_0. The threefold X_0 has nodes at P_0 and Q_0, and is smooth elsewhere along F_ξ.

Let $f\colon X_1 \to X_0$ be the blow up of the points P_0 and Q_0. Denote by E_1^P and E_1^Q be the exceptional divisors of f over the points P_0 and Q_0, respectively. One has

$$E_1^P \cong E_1^Q \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Denote by L_1 the proper transform of F_ξ on X_1. The threefold X_1 is smooth along L_1.

Lemma 4.17. *One has* $\mathcal{N}_{L_1/X_1} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

Proof. As in the proof of Lemmas 4.7 and 4.13, let $\Pi \subset \mathbb{P}^3$ be a general plane containing the line L_ξ, let R be the preimage of Π with respect to the double cover τ, and let R_0 be the proper transform of R on the threefold X_0. Then it follows from equation (4.4) that R_0 has nodes at the points P_0 and Q_0, and is smooth elsewhere. One has

$$K_{R_0} \cdot F_\xi = K_{X_0} \cdot F_\xi = -1.$$

Let R_1 be the proper transform of R_0 on the threefold X_1. Then the surface R_1 is smooth. The morphism $f|_{R_1}\colon R_1 \to R_0$ is the blow up of nodes of R_0, and thus it is crepant. One has $K_{R_1} \cdot L_1 = -1$, so that $L_1^2 = -1$ and

$$\mathcal{N}_{L_1/R_1} \cong \mathcal{O}_{\mathbb{P}^1}(-1).$$

On the other hand, we know that

$$K_{X_1} \cdot L_1 = \big(f^*K_{X_0} + E_1^P + E_1^Q\big) \cdot L_1 = 1,$$

which implies the assertion by Lemma 4.5. $\square$

By Lemma 4.17 and [12, §2], there exists an antiflip $\sigma\colon X_1 \dashrightarrow \check{X}_1$ in the curve L_1. The inverse map σ^{-1} is usually called a Francia flip.

Let $f_\xi\colon U_1 \to \mathbb{P}^2$ be the blow up of the point ξ. Let $Z_1 \cong \mathbb{P}^1$ be the exceptional divisor of the blow up f_ξ, and C_1 be the proper transform of the curve C on U_1. By Proposition 3.2(i), the intersection $C_1 \cap Z_1$ consists of a single point ξ_1, and C_1 has a node at ξ_1. Let $p_1\colon X_1 \dashrightarrow U_1$ and $\check{p}_1\colon \check{X}_1 \dashrightarrow U_1$ be the resulting rational maps. In fact, the rational map $\check{p}_1$ is a morphism. To prove this, we need to recall the explicit construction of σ from [12, §2].

Let $f_1\colon \bar{X}_1 \to X_1$ be the blow up of the curve L_1, and let $\bar{G}_1$ be its exceptional surface. By Lemma 4.17 we have $\bar{G}_1 \cong \mathbb{F}_1$. Denote by $\bar{L}_1$ the unique smooth rational curve in $\bar{G}_1$ such that $\bar{L}_1^2 = -1$.

Lemma 4.18. *One has* $\mathcal{N}_{\bar{L}_1/\bar{X}_1} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Proof. One has

$$\mathcal{N}_{\bar{L}_1/\bar{G}_1} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$$

by construction. On the other hand, we know that $K_{\bar{X}_1} \cdot \bar{L}_1 = 0$, which implies the assertion by Lemma 4.5. $\square$

By Lemma 4.18, we can make an Atiyah flop $\psi\colon \bar{X}_1 \dashrightarrow \hat{X}_1$ in the curve $\bar{L}_1$. Thus, there is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\omega_2"'] \arrow[dr, "\omega_1"] & \\ \hat{X}_1 & & \bar{X}_1, \arrow[ll, dashed, "\psi"'] \end{tikzcd}$$

where ω_1 is the blow up of the curve $\bar{L}_1$, and ω_2 is the contraction of the exceptional divisor $G_W \cong \mathbb{P}^1 \times \mathbb{P}^1$ of ω_1 onto a smooth rational curve $\hat{L}_1$ contained in the smooth

locus of $\hat{X}_1$. Denote by E_W^P, E_W^Q, and $G_{1,W}$ the proper transforms on W of the surfaces E_1^P, E_1^Q and $\bar{G}_1$, respectively.

Let $\hat{G}_1$ be the proper transform of $\bar{G}_1$ on $\hat{X}_1$. Then $\hat{G}_1 \cong \mathbb{P}^2$ and its normal bundle in $\hat{X}_1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-2)$, so that there exists a contraction $g_1\colon \hat{X}_1 \to \check{X}_1$ of the surface $\hat{G}_1$ to a singular point Ξ_1 of type $\frac{1}{2}(1,1,1)$. There is a commutative diagram

(4.19)
$$\begin{tikzcd}
& W \arrow[dl, "\omega_2"'] \arrow[dr, "\omega_1"] & & \\
\hat{X}_1 \arrow[d, "g_1"] & & \bar{X}_1 \arrow[ll, dashed, "\psi"'] \arrow[d, "f_1"] & \\
\check{X}_1 \arrow[dd, dashed, "\check{p}_1"] & & X_1 \arrow[ll, dashed, "\sigma"'] \arrow[dr, "f"] \arrow[ddll, dashed, "p_1"] & \\
& & & X_0 \arrow[d, "\pi"] \\
U_1 \arrow[rrr, "f_\xi"] & & & \mathbb{P}^2.
\end{tikzcd}$$

Lemma 4.20. *The rational map $\check{p}_1$ is a morphism.*

Proof. As in the proof of Lemmas 4.8 and 4.16, let $\mathcal{H}$ be the pencil of curves that are proper transforms on U_1 of lines in $\mathbb{P}^2$ passing through the point ξ. Denote by $\mathcal{H}_{\check{X}_1}$ its proper transform on $\check{X}_1$. To show that $\check{p}_1$ is a morphism, it is enough to show that $\mathcal{H}_{\check{X}_1}$ is base point free. To start with, we show that its proper transform $\mathcal{H}_W$ on W is base point free.

By construction, we know that all base points of the pencil $\mathcal{H}_W$ are contained in the union
$$G_W \cup G_{1,W} \cup E_W^P \cup E_W^Q.$$
Let us show that the pencil $\mathcal{H}_W$ has no base points in these surfaces.

Let l_1 be the class of the ruling of $G_W \cong \mathbb{P}^1 \times \mathbb{P}^1$ that is contracted by ω_2. Since
$$\mathcal{H}_W \sim (\pi \circ f \circ f_1 \circ \omega_1)^* \mathcal{O}_{\mathbb{P}^2}(1) - G_W - G_{1,W} - E_W^P - E_W^Q,$$
we obtain $\mathcal{H}_W|_{G_W} \sim l_1$. Therefore, either two different elements of the pencil $\mathcal{H}_W$ do not have intersection points in G_W, or all of them contain one and the same ruling of class l_1. The latter case is impossible, because the proper transforms of elements of $\mathcal{H}$ on $\bar{X}_1$ are tangent to each other along $\bar{L}_1$ with multiplicity 2. Therefore, the pencil $\mathcal{H}_W$ has no base points in G_W. Also, we have
$$\mathcal{H}_W|_{G_{1,W}} \sim 0,$$
which implies that the surface $G_{1,W}$ is disjoint from a general member of the pencil $\mathcal{H}_W$. In particular, $\mathcal{H}_W$ has no base points in $G_{1,W}$.

Arguing as in the proof of Lemma 4.16, we see that the pencil $\mathcal{H}_W$ does not have base points in the surfaces E_W^P and E_W^Q outside the curves
$$E_W^P \cap G_W, \quad E_W^P \cap G_{1,W}, \quad E_W^Q \cap G_W, \quad E_W^Q \cap G_{1,W}.$$
But we already know that $\mathcal{H}_W$ has no base points in G_W and $G_{1,W}$. This shows that $\mathcal{H}_W$ is base point free. In particular, the rational map $\check{p}_1 \circ g_1 \circ \omega_2$ is a morphism.

Observe that the restrictions

$$\mathcal{H}_W|_{G_W} \sim G_{1,W}|_{G_W} \sim l_1$$

lie in the fibers of the morphism ω_2. Therefore, the proper transform $\mathcal{H}_{\hat{X}_1}$ of the pencil $\mathcal{H}_{\check{X}_1}$ on the threefold $\hat{X}_1$ is base point free, and the surface $\hat{G}_1$ is disjoint from its general member. This shows that $\mathcal{H}_{\check{X}_1}$ is base point free, so that $\check{p}_1$ is a morphism. $\square$

Let us describe the fibers of $\check{p}_1$ over the points of the curve Z_1. Denote by $\bar{E}_1^P$, $\hat{E}_1^P$, and $\check{E}_1^P$ the proper transforms of the surface E_1^P on the threefolds $\bar{X}_1$, $\hat{X}_1$, and $\check{X}_1$, respectively. Similarly, denote by $\bar{E}_1^Q$, $\hat{E}_1^Q$, and $\check{E}_1^Q$ the proper transforms of the surface E_1^Q on the threefolds $\bar{X}_1$, $\hat{X}_1$, and $\check{X}_1$, respectively. One has

$$\check{E}_1^P \cap \check{E}_1^Q = \hat{L}_1.$$

Moreover, the surfaces $\check{E}_1^P$ and $\check{E}_1^Q$ intersect along the curve $\hat{L}_1$, and this intersection is transversal outside the singular point Ξ_1. Furthermore, one has

$$\check{p}_1^{-1}(Z_1) = \check{E}_1^P \cup \check{E}_1^Q.$$

The curve L_1 intersects each of the divisors E_1^P and E_1^Q transversally at a single point. Denote by M_P and M_P' the two rulings of $E_1^P \cong \mathbb{P}^1 \times \mathbb{P}^1$ that pass through the intersection point $L_1 \cap E_1^P$, and denote by M_Q and M_Q' the two rulings of $E_1^Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ that pass through the intersection point $L_1 \cap E_1^Q$. Let $\check{M}_P$, $\check{M}_P'$, $\check{M}_Q$, and $\check{M}_Q'$ be the proper transforms on $\check{X}_1$ of the curves M_P, M_P', M_Q, and M_Q', respectively. Then the curves $\check{M}_P$, $\check{M}_P'$, $\check{M}_Q$, and $\check{M}_Q'$ pass through the singular point Ξ_1, and are mapped by $\check{p}_1$ to the nodal point ξ_1 of the curve C_1. Since

$$K_{\hat{X}_1} \sim_{\mathbb{Q}} g_1^* K_{\check{X}_1} + \frac{1}{2}\hat{G}_1,$$

one has

$$-K_{\check{X}_1} \cdot \check{M}_P = -K_{\check{X}_1} \cdot \check{M}_P' = -K_{\check{X}_1} \cdot \check{M}_Q = -K_{\check{X}_1} \cdot \check{M}_Q' = \frac{1}{2}.$$

This shows that $\check{M}_P + \check{M}_P' + \check{M}_Q + \check{M}_Q'$ is a scheme theoretic fiber of $\check{p}_1$ over ξ_1. All other fibers of $\check{p}_1$ over the points of Z_1 are described by the following remark.

Remark 4.21. The commutative diagram (4.19) gives the commutative diagram

$$\begin{tikzcd} & E_W^P \arrow[dl, "\omega_2"'] \arrow[dr, "\omega_1"] & \\ \hat{E}_1^P \arrow[d, "g_1"'] & & \bar{E}_1^P \arrow[d, "f_1"] \\ \check{E}_1^P \arrow[dr, "\check{p}_1"'] & & E_1^P \arrow[dl, dashed, "p_1"] \\ & Z_1. & \end{tikzcd}$$

Here we denote the restrictions of the morphisms ω_1, ω_2, g_1, f_1, $\check{p}_1$, and the rational map p_1 to the corresponding surfaces by the same symbols for simplicity. The surface E_1^P can be identified with a quadric in $\mathbb{P}^3$, and the rational map p_1 is the linear projection of E_1^P

from a line that is tangent to it at the point $P_1 = L_1 \cap E_1^P$ (cf. the proof of Lemma 4.20). The morphism f_1 is the blow up of the point P_1, the morphism ω_1 is the blow up of the point $\bar{L}_1 \cap \bar{E}_1^P$, the morphism ω_2 is an isomorphism. The morphism g_1 is the contraction of the (-2)-curve $\hat{G}_1 \cap \hat{E}_1^P$ to the node Ξ_1 of the surface $\check{E}_1^P$. By construction, we have $\check{p}_1(\Xi_1) = \xi_1$, and the fiber of $\check{p}_1$ over ξ_1 is $\check{M}_P \cup \check{M}'_P$. The fibers of $\check{p}_1$ over all other points in Z_1 are smooth rational curves. A similar description applies to the surfaces E_1^Q, $\bar{E}_1^Q$, E_W^Q, $\hat{E}_1^Q$, and $\check{E}_1^Q$.

Let $\hat{M}_P$, $\hat{M}'_P$, $\hat{M}_Q$, and $\hat{M}'_Q$ be the proper transforms on $\hat{X}_1$ of the curves M_P, M'_P, M_Q, and M'_Q, respectively. The curves $\hat{M}_P$, $\hat{M}'_P$, $\hat{M}_Q$, and $\hat{M}'_Q$ are pairwise disjoint, and each of them is disjoint from the curve $\hat{L}_1$.

Lemma 4.22. *Each of the normal bundles $\mathcal{N}_{\hat{M}_P/\hat{X}_1}$, $\mathcal{N}_{\hat{M}'_P/\hat{X}_1}$, $\mathcal{N}_{\hat{M}_Q/\hat{X}_1}$, and $\mathcal{N}_{\hat{M}'_Q/\hat{X}_1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.*

Proof. Let $\bar{M}_P$, $\bar{M}'_P$, $\bar{M}_Q$, and $\bar{M}'_Q$ be the proper transforms on $\bar{X}_1$ of the curves M_P, M'_P, M_Q, and M'_Q, respectively. Then $\bar{M}_P$, $\bar{M}'_P$, $\bar{M}_Q$, and $\bar{M}'_Q$ are disjoint from the curve $\bar{L}_1$. Therefore, to prove the assertion of the lemma, it is enough to compute the normal bundles of the latter four curves on $\bar{X}_1$.

One has $\mathcal{N}_{\bar{M}_P/\bar{E}_1^P} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. On the other hand, we compute $K_{\bar{X}_1} \cdot \bar{M}_P = 0$, so that the assertion for the curve $\bar{M}_P$ follows from Lemma 4.5. For the curves $\bar{M}'_P$, $\bar{M}_Q$ and $\bar{M}'_Q$ the argument is similar. ∎

By Lemma 4.22, we can make simultaneous Atiyah flops in the curves $\hat{M}_P$, $\hat{M}'_P$, $\hat{M}_Q$, and $\hat{M}'_Q$. Let $\phi\colon \hat{X}_1 \dashrightarrow \bar{X}_2$ be the composition of these four flops.

Let $\hat{\omega}_1\colon \hat{W} \to \hat{X}_1$ be the blow up of the curves $\hat{M}_P$, $\hat{M}'_P$, $\hat{M}_Q$, and $\hat{M}'_Q$. Denote by N_P, N'_P, N_Q, and N'_Q the exceptional surfaces of $\hat{\omega}_1$ that are mapped to the curves $\hat{M}_P$, $\hat{M}'_P$, $\hat{M}_Q$, and $\hat{M}'_Q$, respectively. Then there is a commutative diagram

$$\begin{tikzcd} & \hat{W} \arrow[dl, "\hat{\omega}_2"'] \arrow[dr, "\hat{\omega}_1"] & \\ \bar{X}_2 & & \hat{X}_1, \arrow[ll, dashed, "\phi"'] \end{tikzcd}$$

where $\hat{\omega}_2$ is the contraction of the surfaces N_P, N'_P, N_Q, and N'_Q to smooth rational curves contained in the smooth locus of $\bar{X}_2$.

Let $f_{\xi_1}\colon U_2 \to U_1$ be the blow up of the point ξ_1. Denote by T_2 the exceptional divisor of the blow up f_{ξ_1}, and by Z_2 and C_2 the proper transforms of the curves Z_1 and C on the surface U_2, respectively. Note that the curves C_2 and Z_2 are disjoint. Furthermore, let $\bar{p}_2\colon \bar{X}_2 \dashrightarrow U_2$ be the resulting rational map. We have constructed the following

commutative diagram

(4.23)
$$
\begin{tikzcd}
& \hat{W} \arrow[dl, "\hat{\omega}_2"'] \arrow[dr, "\hat{\omega}_1"] & & \\
\bar{X}_2 \arrow[dd, dashed, "\bar{p}_2"] & & \hat{X}_1 \arrow[ll, dashed, "\phi"'] \arrow[dr, "g_1"] & \\
& & & \check{X}_1 \arrow[d, "\check{p}_1"] \\
U_2 \arrow[rrr, "f_{\xi_1}"] & & & U_1.
\end{tikzcd}
$$

Lemma 4.24. *The rational map $\bar{p}_2$ is a morphism.*

Proof. Let $\mathcal{B}_{U_1}$ be the linear subsystem in $|f_\xi^*\mathcal{O}_{\mathbb{P}^2}(2) - Z_1|$ consisting of all curves that pass through the point ξ_1. Note that the base locus of $\mathcal{B}_{U_1}$ is the point ξ_1. Moreover, the point ξ_1 is a scheme theoretic intersection of curves in $\mathcal{B}_{U_1}$. Denote by $\mathcal{B}_{U_2}$ the proper transform of $\mathcal{B}_{U_1}$ on U_2, so that $\mathcal{B}_{U_2}$ is a base point free linear system.

Denote by $\mathcal{B}_{\check{X}_1}$ the proper transform of $\mathcal{B}_{U_2}$ on $\check{X}_1$ via $\check{p}_1$. Then the base locus of $\mathcal{B}_{\check{X}_1}$ consists of the curves $\check{M}_P$, $\check{M}'_P$, $\check{M}_Q$, and $\check{M}'_Q$. Moreover, the union of these curves is a scheme theoretic intersection of surfaces in $\mathcal{B}_{\check{X}_1}$.

Denote by $\mathcal{B}_{\bar{X}_2}$ the proper transform of $\mathcal{B}_{\check{X}_1}$ on the threefold $\bar{X}_2$. To prove that $\bar{p}_2$ is a morphism, it is enough to show that $\mathcal{B}_{\bar{X}_2}$ is base point free. Denote by $\mathcal{B}_{\bar{X}_1}$, $\mathcal{B}_{\hat{X}_1}$, and $\mathcal{B}_{\hat{W}}$ the proper transforms of $\mathcal{B}_{\check{X}_1}$ on the threefolds $\bar{X}_1$, $\hat{X}_1$, and $\hat{W}$, respectively. To show that $\mathcal{B}_{\bar{X}_2}$ is base point free, let us describe the base loci of $\mathcal{B}_{\bar{X}_1}$, $\mathcal{B}_{\hat{X}_1}$, and $\mathcal{B}_{\hat{W}}$.

We claim that the base locus of $\mathcal{B}_{\bar{X}_1}$ consists of the curves $\bar{M}_P$, $\bar{M}'_P$, $\bar{M}_Q$, and $\bar{M}'_Q$. We already know that these curves are contained in the base locus. On the other hand, the base locus of $\mathcal{B}_{\hat{X}_1}$ is contained in the union of the curves $\hat{M}_P$, $\hat{M}'_P$, $\hat{M}_Q$, and $\hat{M}'_Q$, and the surface $\hat{G}_1$. Thus, the base locus of $\mathcal{B}_{\bar{X}_1}$ consists of the curves $\bar{M}_P$, $\bar{M}'_P$, $\bar{M}_Q$, and $\bar{M}'_Q$, and a (possibly empty) subset of $\bar{G}_1$. Using Lemma 3.9, we obtain the equivalence

$$\mathcal{B}_{\bar{X}_1} \sim (\pi \circ f \circ f_1)^*\,\mathcal{O}_{\mathbb{P}^2}(2) - 2\bar{G}_1 - \bar{E}_1^P - \bar{E}_1^Q.$$

This gives

$$\mathcal{B}_{\bar{X}_1}|_{\bar{G}_1} \sim 2\bar{L}_1 + 2t,$$

where t is the class of a ruling of $\bar{G}_1 \cong \mathbb{F}_1$. The latter equivalence together with Lemma 3.9 shows that the restriction $\mathcal{B}_{\bar{X}_1}|_{\bar{G}_1}$ does not have base curves that are mapped dominantly to the curve L_1 by f_1. In particular, a general surface in $\mathcal{B}_{\bar{X}_1}$ is disjoint from the curve $\bar{L}_1$. On the other hand, the four points

$$\bar{M}_P \cap \bar{G}_1, \quad \bar{M}'_P \cap \bar{G}_1, \quad \bar{M}_Q \cap \bar{G}_1, \quad \bar{M}'_Q \cap \bar{G}_1$$

are contained in the base locus of the restriction $\mathcal{B}_{\bar{X}_1}|_{\bar{G}_1}$. This implies that $\mathcal{B}_{\bar{X}_1}|_{\bar{G}_1}$ is a pencil whose base locus consists of exactly these four points. In particular, the base locus of $\mathcal{B}_{\bar{X}_1}$ consists of the curves $\bar{M}_P$, $\bar{M}'_P$, $\bar{M}_Q$, and $\bar{M}'_Q$.

Since a general surface in $\mathcal{B}_{\bar{X}_1}$ is disjoint from $\bar{L}_1$, we see that the base locus of $\mathcal{B}_{\hat{X}_1}$ consists of the curves $\hat{M}_P$, $\hat{M}'_P$, $\hat{M}_Q$, and $\hat{M}'_Q$. By the same reason, we see that the

restriction of $\mathcal{B}_{\hat{X}_1}$ to $\hat{G}_1 \cong \mathbb{P}^2$ is a pencil of conics that pass through the four points

$$\hat{M}_P \cap \hat{G}_1, \quad \hat{M}'_P \cap \hat{G}_1, \quad \hat{M}_Q \cap \hat{G}_1, \quad \hat{M}'_Q \cap \hat{G}_1.$$

In particular, these four points are in general position.

Computing the classes of the restrictions of the linear system $\mathcal{B}_{\hat{W}}$ to the exceptional divisors N_P, N'_P, N_Q, and N'_Q, we see that they all lie in the fibers of $\hat{\omega}_2$. This shows that both linear systems $\mathcal{B}_{\hat{W}}$ and $\mathcal{B}_{\bar{X}_2}$ are base point free. Thus, we proved that $\bar{p}_2$ is a morphism. $\square$

Let $\bar{G}_2$, $\bar{E}_2^P$, and $\bar{E}_2^Q$ be the proper transforms on $\bar{X}_2$ of the surfaces $\hat{G}_1$, $\hat{E}_1^P$, and $\hat{E}_1^Q$, respectively. Then $\bar{E}_2^P$ (respectively, $\bar{E}_2^Q$) is isomorphic to $\mathbb{F}_1$ since it is obtained from the surface $\hat{E}_1^P$ (respectively, $\hat{E}_1^Q$) by blowing down two (-1)-curves $\hat{M}_P$ and $\hat{M}'_P$ (respectively, $\hat{M}_Q$ and $\hat{M}'_Q$) as a result of flopping them (cf. Remark 4.21). Similarly, $\bar{G}_2$ is a smooth del Pezzo surface of degree 5 since it is obtained from the surface $\hat{G}_1 \cong \mathbb{P}^2$ by blowing up four points

$$\hat{M}_P \cap \hat{G}_1, \quad \hat{M}'_P \cap \hat{G}_1, \quad \hat{M}_Q \cap \hat{G}_1, \quad \hat{M}'_Q \cap \hat{G}_1,$$

which are in general position (see the proof of Lemma 4.24).

We know that the preimage of Z_2 via $\bar{p}_2$ is the union of the surfaces $\bar{E}_2^P$ and $\bar{E}_2^Q$. These surfaces intersect transversally along the curve that is a unique (-1)-curve on each of them. Moreover, the restrictions

$$\bar{p}_2|_{\bar{E}_2^P} \colon \bar{E}_2^P \to Z_2$$

and

$$\bar{p}_2|_{\bar{E}_2^Q} \colon \bar{E}_2^Q \to Z_2$$

are just natural projections of $\bar{E}_2^P \cong \mathbb{F}_1$ and $\bar{E}_2^Q \cong \mathbb{F}_1$ to $\mathbb{P}^1$. In particular, the curve Z_2 is contained in the degeneration curve of the conic bundle $\bar{p}_2$.

By construction, the preimage of the curve T_2 via $\bar{p}_2$ is the surface $\bar{G}_2$, and the induced morphism $\bar{p}_2|_{\bar{G}_2} \colon \bar{G}_2 \to T_2$ is a conic bundle with three reducible fibers. In particular, the curve T_2 is not contained in the degeneration curve of the conic bundle $\bar{p}_2$, so that the latter degeneration curve is $Z_2 \cup C_2$. Note that the fibers of $\bar{p}_2$ over the two points in $C_2 \cap T_2$ must be reducible. Also the fiber of $\bar{p}_2$ over the point $T_2 \cap Z_2$ is reducible. Thus, these three fibers are all reducible fibers of $\bar{p}_2$ over T_2.

There are contractions

$$f_2^P \colon \bar{X}_2 \to X_2^P$$

and

$$f_2^Q \colon \bar{X}_2 \to X_2^Q$$

of the surfaces $\bar{E}_2^P$ and $\bar{E}_2^Q$ to the curves contained in the smooth loci of the threefolds X_2^P and X_2^Q, respectively. Let $p_2^P \colon X_2^P \to U_2$ and $p_2^Q \colon X_2^Q \to U_2$ be the resulting morphisms. Then there is a commutative diagram

(4.25)
$$\begin{tikzcd}
 & \bar{X}_2 \arrow[dl, "f_2^Q"'] \arrow[dr, "f_2^P"] \arrow[dd, "\bar{p}_2"] & \\
X_2^Q \arrow[dr, "p_2^Q"'] & & X_2^P \arrow[dl, "p_2^P"] \\
 & U_2. &
\end{tikzcd}$$

Corollary 4.26. *The curve C_2 is the degeneration curve of both conic bundles p_2^P and p_2^Q.*

Gluing together the commutative diagrams (4.19), (4.23), and (4.25), we obtain a commutative diagram

$$
\begin{tikzcd}
 & & \hat{W} \arrow[dl, "\hat{\omega}_2"'] \arrow[dr, "\hat{\omega}_1"] & & W \arrow[dl, "\omega_2"'] \arrow[dr, "\omega_1"] & & & \\
 & \bar{X}_2 \arrow[ddl, "f_2^Q"'] \arrow[dddd, "\bar{p}_2"] \arrow[ddrr, "f_2^P"] & & \hat{X}_1 \arrow[ll, dashed, "\phi"'] \arrow[drr, "g_1"] & & \bar{X}_1 \arrow[ll, dashed, "\psi"'] \arrow[dr, "f_1"] & & \\
 & & & & & \check{X}_1 \arrow[ddd, "\check{p}_1"] & X_1 \arrow[l, dashed, "\sigma"'] \arrow[dddl, dashed, "p_1"] \arrow[ddr, "f"] & \\
X_2^Q \arrow[ddr, "p_2^Q"'] & & & X_2^P \arrow[ddll, "p_2^P"] & & & & \\
 & & & & & & & X_0 \arrow[d, "\pi"] \\
 & U_2 \arrow[rrrr, "f_{\xi_1}"] & & & & U_1 \arrow[rr, "f_{\xi}"] & & \mathbb{P}^2.
\end{tikzcd}
$$

. Now we are ready to finish the proof.

Proof of Theorem 4.2. Note that Constructions I, II, and III are local over the base of the conic bundle. Thus, they are applicable not only to the conic bundle π but also to any other conic bundle which is obtained from $\pi\colon X_0 \to \mathbb{P}^2$ by a birational transformation that is local over the base, provided that they are carried out over neighborhoods of points of the base not influenced by this transformation.

We start with the conic bundle $\pi\colon X_0 \to \mathbb{P}^2$. Keeping in mind Proposition 3.2 and applying Construction I in neighborhoods of points of $\mathbb{P}^2$ where the curve C has a node and over which the fiber of π contains a singular point of X_0, we obtain the birational morphism $\varrho_n\colon U_n \to \mathbb{P}^2$. Applying Construction II in neighborhoods of points of U_n where the proper transform of C has cusps, we obtain the birational morphism $\varrho_c\colon U_c \to U_n$. Finally, applying Construction III in neighborhoods of points of U_c where the proper transform of C has tacnodes, we obtain the birational morphisms $\varrho_t\colon U_t \to U_c$ and $\varrho_t'\colon U \to U_t$. We put

$$\varrho = \varrho_n \circ \varrho_c \circ \varrho_t \circ \varrho_t'.$$

We denote by $\rho\colon V \dashrightarrow X_0$ the birational map provided by Constructions I, II, and III. Choosing the conic bundle $p_2\colon X_2 \to U_2$ after performing Constructions I or II, and choosing any of the conic bundles $p_2^P\colon X_2^P \to U_2$ or $p_2^Q\colon X_2^Q \to U_2$ after performing Construction III, we finally obtain a conic bundle $\nu\colon V \to U$. This completes the diagram (4.1).

We already know that the threefold V and the surface U are smooth. Also, we know from Lefschetz theorem that $\operatorname{rk}\operatorname{Pic}(X) = 1$, so that $\operatorname{rk}\operatorname{Cl}(X) = 1$ by $\mathbb{Q}$-factoriality assumption. Keeping track of the blow ups we make in course of our construction, we see that at the starting point we have an equality $\operatorname{rk}\operatorname{Cl}(X_0/\mathbb{P}^2) = 1$, and Constructions I, II and III preserve this equality. At the end of the day we arrive to smooth varieties V

and U, and thus conlcude that

$$\mathrm{rk}\ \mathrm{Pic}(V/U) = \mathrm{rk}\ \mathrm{Cl}(V/U) = 1.$$

This means that ν is a standard conic bundle. Since the divisor $-K_V$ is ν-ample and U is a projective surface, we see that V is also projective (although a result of a flop may a priori be not projective).

Other assertions of the theorem hold by construction. □

Constructions I, II, and III are analogues of the constructions in the proof of [19, Proposition 2.4].

5. Irrational quartic double solids

The main goal of this section is to prove Theorem 1.2. Another goal is to show that Conjecture 1.9 follows from Conjecture 2.2. To achieve these goals, we need the following straightforward result.

Proposition 5.1. *Let U be a smooth surface, and let Δ be a reduced curve in U. Suppose that*

$$\Delta \sim -2K_U,$$

the curve Δ is not a smooth rational curve, and Δ satisfies conditions (A) *and* (B) *in Corollary 2.1. Then $-K_U$ is numerically effective (nef).*

Proof. Suppose that $-K_U$ is not nef. Then there is an irreducible curve $\Delta_1 \subset U$ such that $\Delta \cdot \Delta_1 < 0$. This, in particular, means that Δ_1 is an irreducible component of Δ.

We claim that Δ is a reducible curve. Indeed, if $\Delta = \Delta_1$, then $\Delta^2 < 0$. On the other hand, the adjunction formula implies that

$$2p_a(\Delta) - 2 = \Delta^2 + K_U \cdot \Delta = \Delta^2 - \frac{\Delta^2}{2} = \frac{\Delta^2}{2},$$

where $p_a(\Delta) = 1 - \chi(\mathcal{O}_\Delta)$ is the arithmetic genus of Δ. Thus, if Δ is irreducible, then its arithmetic genus must be zero, so that Δ is a smooth rational curve. The latter is impossible, because Δ satisfies condition (B) of Corollary 2.1.

We see that Δ is reducible, and Δ_1 is its irreducible component. Denote the union of its remaining irreducible components by Δ_2, so that $\Delta = \Delta_1 \cup \Delta_2$. Then

$$0 > \Delta \cdot \Delta_1 = \Delta_1^2 + \Delta_1 \cdot \Delta_2.$$

On the other hand, the adjunction formula gives

$$2p_a(\Delta_1) - 2 = \Delta_1^2 + K_U \cdot \Delta_1 = \Delta_1^2 - \left(\frac{\Delta_1 + \Delta_2}{2}\right) \cdot \Delta_1 = \frac{\Delta_1^2}{2} - \frac{\Delta_1 \cdot \Delta_2}{2},$$

so that

$$4p_a(\Delta_1) - 4 = \Delta_1^2 - \Delta_1 \cdot \Delta_2.$$

Thus, if $p_a(\Delta_1) > 0$, then

$$0 \leqslant 4p_a(\Delta_1) - 4 = \Delta_1^2 - \Delta_1 \cdot \Delta_2 < -2\Delta_1 \cdot \Delta_2,$$

which gives a contradiction with $\Delta_1 \cdot \Delta_2 \geqslant 0$. Hence, we have $p_a(\Delta_1) = 0$, which implies that Δ_1 is a smooth rational curve. Therefore

$$-4 = \Delta_1^2 - \Delta_1 \cdot \Delta_2$$

by the adjunction formula. Since $\Delta_1^2 + \Delta_1 \cdot \Delta_2 < 0$ by assumption, we have $\Delta_1^2 < -2$. Thus

$$-4 = \Delta_1^2 - \Delta_1 \cdot \Delta_2 < -2 - \Delta_1 \cdot \Delta_2,$$

which gives $\Delta_1 \cdot \Delta_2 \leqslant 1$. This is impossible, because Δ satisfies conditions (A) and (B) of Corollary 2.1. □

Now let $\tau\colon X \to \mathbb{P}^3$ be a double cover branched over a nodal quartic surface S. To prove Theorem 1.2 and to show that Conjecture 1.9 follows from Conjecture 2.2, we must prove that X is irrational in each of the following cases:

- when S has at most six nodes;
- or when X is $\mathbb{Q}$-factorial and Conjecture 2.2 is true.

If S is smooth, then X is irrational by [25, Corollary 4.7(b)]. Thus, we may assume that S is singular. If S has at most five nodes, then X is $\mathbb{Q}$-factorial by Theorem 1.8. Similarly, if S has exactly six nodes, then X is $\mathbb{Q}$-factorial by Theorem 1.8 with the only exception when X is birational to a smooth cubic threefold in $\mathbb{P}^4$, and thus irrational by [6, Theorem 13.12]. Therefore, we may also assume that X is $\mathbb{Q}$-factorial. Thus, we can apply all results of Sections 3 and 4 to X.

By Theorem 4.2, the threefold X is birational to a smooth threefold V with a structure of a standard conic bundle $\nu\colon V \to U$ and there exists a birational morphism $\varrho\colon U \to \mathbb{P}^2$ that is a composition of $|\mathrm{Sing}(S)| - 1$ blow ups. Denote by Δ the degeneration curve of the conic bundle ν. In particular, there exists a pair (Δ', I) of a connected nodal curve Δ' and an involution I on it such that $\Delta \cong \Delta'/I$, the nodes of Δ' are exactly the fixed points of I, and I does not interchange branches at these points. One has $\Delta \sim -2K_U$ by Theorem 4.2(iii).

Corollary 5.2. *Conjecture 2.2 implies Conjecture 1.9.*

Proof. We have $2K_U + \Delta \sim 0$, so that the linear system $|2K_U + \Delta|$ is not empty. Thus, the irrationality of X follows from Conjecture 2.2. □

By Corollary 2.1, the curve Δ satisfies its conditions (A) and (B). Thus, $-K_U$ is nef by Proposition 5.1. Put $d = K_U^2$. Then

$$d = 10 - \big|\mathrm{Sing}(S)\big|$$

by Theorem 4.2(i).

Now we are ready to prove Theorem 1.2. Until the end of the section we assume that S has at most six nodes, so that $d \geqslant 4$. In particular, U is a *weak del Pezzo surface* (see [9]).

Lemma 5.3. *The curve Δ is connected.*

Proof. Since $-K_U$ is nef and big, we have $h^1(\mathcal{O}_U(-2K_U)) = 0$ by the Kawamata–Viehweg vanishing theorem (see [14]). This implies connectedness of Δ, because $\Delta \sim -2K_U$. □

We plan to apply Theorem 2.6 to V. Unfortunately, the curve Δ may not satisfy condition (S). Luckily, we can explicitly describe each case when Δ does not satisfy it. This description is given by the following three lemmas.

Lemma 5.4. *Let E be a (-2)-curve on U. Then either E is contracted by ϱ to a point, or $\varrho(E)$ is a line in $\mathbb{P}^2$. Moreover, either E is disjoint from Δ, or E is an irreducible component of Δ. Furthermore, if E is an irreducible component of Δ, then it intersects the curve $\Delta - E$ by two points. In particular, if Δ satisfies condition* (S) *of Theorem 2.6, then E is disjoint from Δ.*

Proof. If E is not contracted by ϱ to a point, then $\varrho(E)$ is a line, because $d \geqslant 4$. If E is not an irreducible component of the curve Δ, then

$$\Delta \cdot E = -2K_U \cdot E = 0,$$

which implies that E is disjoint from Δ. If E is an irreducible component of Δ, then

$$(\Delta - E) \cdot E = (-2K_U - E) \cdot E = 2.$$

Since Δ is nodal, this means that $\Delta - E$ intersects E by two points. In particular, Δ does not satisfy condition (S) of Theorem 2.6 in this case. $\square$

Lemma 5.5. *At most two irreducible components of the curve Δ are (-2)-curves. Moreover, all other (-2)-curves on U are disjoint from Δ. Furthermore, for the curve Δ we have only the following possibilities:*

- *the curve Δ contains a unique (-2)-curve E in U and*
$$\Delta = E + \Omega,$$
where Ω is a nodal curve such that $\varrho(\Omega)$ is a (possibly reducible) quintic curve, $\varrho(E)$ is a line, and $E \cap \Omega$ consists of two points;
- *the curve Δ contains two (-2)-curves E_1 and E_2 in U, the curves E_1 and E_2 are disjoint, and*
$$\Delta = E_1 + E_2 + \Upsilon,$$
where Υ is a nodal curve such that $\varrho(\Upsilon)$ is a (possibly reducible) quartic curve, $\varrho(E_1)$ and $\varrho(E_1)$ are lines, and each intersection $E_1 \cap \Upsilon$ or $E_1 \cap \Upsilon$ consists of two points.

Proof. Suppose that at least three irreducible components of Δ are (-2)-curves. Denote them by E_1, E_2, and E_3. Then ϱ maps them to lines in $\mathbb{P}^2$ by Lemma 5.4. This implies that ϱ blows up at least six points on these lines, which is impossible, because we assume that $d \geqslant 4$. This shows that at most two irreducible components of the curve Δ are (-2)-curves. All remaining assertions easily follow from Lemma 5.4. $\square$

Lemma 5.6. *Suppose that Δ does not satisfy condition* (S) *of Theorem 2.6, i. e. there is a splitting $\Delta = \Delta_1 \cup \Delta_2$ such that $\Delta_1 \cdot \Delta_2 = 2$. Then either Δ_1 or Δ_2 is a (-2)-curve.*

Proof. We claim that Δ_1 and Δ_2 are linearly independent in $\mathrm{Pic}(S) \otimes \mathbb{Q}$. Indeed, if $\Delta_1 \sim_{\mathbb{Q}} \lambda\Delta_2$ for some rational number λ, then

$$2 = \Delta_1 \cdot \Delta_2 = \lambda\Delta_2^2,$$

which means that $\lambda = 1$ or $\lambda = 2$. Moreover, one has

$$-K_U \sim_{\mathbb{Q}} \frac{\lambda+1}{2}\Delta_2$$

and

$$4 \leqslant (-K_U)^2 = \left(\frac{\lambda+1}{2}\right)^2 \Delta_2^2 = \frac{2}{\lambda}\left(\frac{\lambda+1}{2}\right)^2,$$

which is impossible for $\lambda = 1$ or $\lambda = 2$.

Applying the Hodge index Theorem, we get

$$\begin{vmatrix} \Delta_1^2 & \Delta_1\Delta_2 \\ \Delta_1\Delta_2 & \Delta_1^2 \end{vmatrix} < 0,$$

which means that

$$\Delta_1^2\Delta_2^2 < (\Delta_1\Delta_2)^2 = 4. \tag{5.7}$$

We claim that an arithmetic genus of either Δ_1 or Δ_2 is non-positive. Indeed, otherwise

$$0 \leqslant 2p_a(\Delta_i) - 2 = \Delta_i(\Delta_i + K_U) = \frac{\Delta_i^2}{2} - \frac{\Delta_1\Delta_2}{2} = \frac{\Delta_i^2}{2} - 1.$$

This means that $\Delta_i^2 \geqslant 2$, and thus $\Delta_1^2\Delta_2^2 \geqslant 4$, which contradicts (5.7).

Without loss of generality, we may assume that $p_a(\Delta_1) \leqslant 0$. Then $\Delta_1^2 \leqslant -2$ by the adjunction formula. Thus, if Δ_1 is irreducible, then we are done. Therefore, we assume that Δ_1 has at least two irreducible components. Then the degree of the curve $\varrho(\Delta_1)$ is at least two, and thus the degree of the curve $\varrho(\Delta_2)$ is at most four. On the other hand, we have

$$\Delta_2^2 \geqslant \Delta_2^2 + \Delta_1^2 + 2 = (\Delta_1 + \Delta_2)^2 - 2 = (-2K_U)^2 - 2 = 4d - 2 \geqslant 14,$$

which implies that the degree of the curve $\varrho(\Delta_2)$ is exactly four, Δ_1 has exactly two irreducible components, and each of these components is mapped by ϱ to a line in $\mathbb{P}^2$. This is impossible, because

$$2 = \Delta_1 \cdot \Delta_2 \geqslant \varrho(\Delta_1) \cdot \varrho(\Delta_2) - (9 - d) = d - 1 \geqslant 3,$$

which is absurd. □

Since $d \geqslant 4$, the linear system $|-K_U|$ is base point free and gives a morphism $\phi\colon U \to \mathbb{P}^d$. Denote by Y the image of U via ϕ. Then U is a del Pezzo surface with du Val singularities, and ϕ induces a birational morphism $\varphi\colon U \to Y$ that contracts all (-2)-curves on U to singular points of the surface Y. Since $d \geqslant 4$, the surface Y is an intersection of quadrics in $\mathbb{P}^d$, see, for example, [9].

Put $\nabla = \varphi(\Delta)$. Then $\nabla \in |-2K_Y|$. By Lemma 5.5, the curve ∇ is connected and nodal. Moreover, it follows from Lemma 5.5 and Remark 2.9 that there exists a pair (∇', J) of a connected nodal curve ∇' and an involution J on it such that $\nabla \cong \nabla'/J$, the nodes of ∇' are exactly the fixed points of J, the involution J does not interchange branches at these points, and

$$\operatorname{Prym}(\Delta', I) \cong \operatorname{Prym}(\nabla', J)\,.$$

Since U is rational, the exact sequence

$$0 \to \mathcal{O}_U \to \omega_U \otimes \mathcal{O}_U(\Delta) \to \omega_\Delta \to 0$$

implies that there is a surjection

$$H^0(-K_U) = H^0(K_U + \Delta) \twoheadrightarrow H^0(K_\Delta),$$

so for the anticanonical map ϕ one has

$$\phi|_\Delta = \kappa_\Delta,$$

where κ_Δ is a canonical map of the curve Δ. Therefore, we see that ∇ is a connected nodal curve canonically embedded into $\mathbb{P}^d$, which is an intersection of quadrics. In particular, ∇ is not hyperelliptic by Remark 2.5. Moreover, for every splitting $\nabla = \nabla_1 \cup \nabla_2$, the intersection $\nabla_1 \cap \nabla_2$ consists of at least 4 points. This follows from Remark 2.1 and Lemmas 5.5 and 5.6. Thus, $\operatorname{Prym}(\nabla', J)$ is not a sum of Jacobians of smooth curves by Corollary 2.7. On the other hand, Theorem 2.4 implies that

$$\mathrm{J}(V) \cong \operatorname{Prym}(\Delta', I) \cong \operatorname{Prym}(\nabla', J)\,,$$

where $\mathrm{J}(V)$ is the intermediate Jacobian of the threefold V. This shows that V is irrational by [6, Corollary 3.26] and completes the proof of Theorem 1.2.

6. Rational quartic double solids

In this section, we prove Theorem 1.5. Let $\tau\colon X \to \mathbb{P}^3$ be a double cover branched over a nodal quartic surface S. Suppose that X is not $\mathbb{Q}$-factorial. We are going to show that X is rational unless it is described by Example 1.4.

Let $f\colon X' \to X$ be a $\mathbb{Q}$-factorialization of the threefold X (see [15, Corollary 4.5]). Then

$$-K_{X'} \sim f^*\left(\tau^*\left(\mathcal{O}_{\mathbb{P}^3}(2)\right)\right), \tag{6.1}$$

and X' has at most nodes as singularities.

Since $K_{X'}$ is not nef, the cone $\overline{\mathrm{NE}}(X')$ has an extremal ray that has negative intersection with $K_{X'}$. Let $\eta\colon X' \to Y$ be a contraction of this extremal ray (see [22, Corollary 2.9]). Since X is not $\mathbb{Q}$-factorial, Y is not a point. If η is a conic bundle, then (6.1) implies that the pull-back of a plane in $\mathbb{P}^3$ via $\tau \circ f$ is a section of η, so that X is rational. Similarly, if η is a del Pezzo fibration, then (6.1) implies that the canonical class of its general fiber is divisible by two in the Picard group, so that the general fiber of η is a quadric surface by the adjunction formula, and X is rational in this case as well. Hence, to complete the proof, we may assume that η is birational. Then it follows from (6.1) and [7, Theorem 5] that η is a blow up of a smooth point in Y. Let us denote this point by P.

The divisor $-K_Y$ is nef by (6.1). Moreover, we have

$$(-K_Y)^3 = (-K_{X'})^3 + 8 = (-K_X)^3 + 8 = 24,$$

which implies that $-K_Y$ is big. By [22, Theorem 2.1], the linear system $|-nK_Y|$ is base point free for some $n > 0$, and it gives a birational morphism $\phi\colon Y \to Z$ such that Z is a Fano threefold with canonical Gorenstein singularities. Moreover, [22, Theorem 2.1] also implies that $-K_Z$ is divisible by 2 in $\mathrm{Pic}(Z)$. Since

$$(-K_Z)^3 = (-K_Y)^3 = 24,$$

the threefold Z must be isomorphic to a cubic threefold in $\mathbb{P}^4$ (see, for example, [17, Theorem 3.4]). In particular, if Z is singular, then X is rational. Thus, to complete the proof, we may assume that Z is smooth. Then ϕ is an isomorphism, so we may assume that $Z = Y$. Therefore, there is a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\eta"] \arrow[d, "f"'] & Y \arrow[d, dashed, "\gamma"] \\ X \arrow[r, "\tau"] & \mathbb{P}^3 \end{tikzcd}$$

where γ is a linear projection from the point P. Since X is nodal, the cubic Y contains exactly six lines that pass through P, and f is the contraction of their proper transforms. This means that X is described by Example 1.4.

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